Special birational structures on non-Kählerian complex surfaces

Georges Dloussky
Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373,
39, rue F.Joliot-Curie, 13453 Marseille, France
georges.dloussky@univ-amu.fr

Abstract

We investigate the following conjecture: all compact non-Kählerian complex surfaces admit birational structures. After Inoue-Kobayashi-Ochiai, the remaining cases to study are essentially surfaces in class VII_0^+. We show that Kato surfaces with a cycle and only one branch of rational curves admit a special birational structure given by new normal forms of contracting germs in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$. In particular all surfaces S with GSS and second Betti number satisfying $0 \leq b_2(S) \leq 3$ admit a birational structure. From the existence of a special birational structure we deduce a developing meromorphic mappings $\tilde{S} \to \mathbb{P}^2(\mathbb{C})$ from the universal cover of S to $\mathbb{P}^2(\mathbb{C})$ which blows down an infinite number of rational curves. From this mapping we recover a GSS.

Résumé

On étudie la conjecture suivante: toute surface complexe compacte non kählerienne admet une structure birationnelle. D'après Inoue-Kobayashi-Ochiai, les cas restant à étudier sont essentiellement les surfaces de la classe VII_0^+. On démontre que les surfaces de Kato qui ont un cycle avec un seul arbre de courbes rationnelles admettent une structure birationnelle spéciale définie par de nouvelles formes normales de germes de contractions dans le groupe de Cremona $Bir(\mathbb{P}^2(\mathbb{C}))$. En particulier toute surface S contenant une coquille sphérique globale (CSG) et pour laquelle le second nombre de Betti vérifie $0 \leq b_2(S) \leq 3$ admet une structure birationnelle. De l'existence d'une structure birationnelle on déduit une application méromorphe développante $\tilde{S} \to \mathbb{P}^2(\mathbb{C})$ du revêtement universel de S dans $\mathbb{P}^2(\mathbb{C})$ qui écrase une infinité de courbes rationnelles. Cette application permet de reconstituer la CSG.

Keywords: Complex surface, non-Kähler, G-structure, birational structure.

MSC codes: 14E05, 14E07, 32J15, 32Q57

1 Introduction

What is the best G-structure on a compact manifold ? The classification of Inoue-Kobayashi-Ochiai [14, 18] shows that all compact complex non-Kählerian surfaces but some Hopf surfaces and surfaces in class VII_0^+ (i.e. in class VII_0 with $b_2 > 0$) admit affine structures. In view of the explicit construction of Kato surfaces (i.e. minimal surfaces S containing a global spherical shell (GSS) with $b_2(S) > 0$) and the particular cases of Enoki surfaces and Inoue-Hirzebruch surfaces the best G-structure should be obtained for a subgroup of $Bir(\mathbb{P}^2(\mathbb{C}))$. It justifies the following
Conjecture: All compact complex non-Kähler surfaces admit birational structures.
The conjecture is clearly satisfied for all Hopf surfaces because they are defined by an invertible contracting polynomial mapping. Remains the case of surfaces in class VII_0^+. Since the only known surfaces in class VII_0^+ are Kato surfaces and since it is conjectured that there are no others, this article focuses on the following problem: Do compact surfaces with GSS admit a birational structure, i.e. is there an atlas with transition mappings in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$. As stronger

requirement, is there in each conjugation class of contracting germs of the form $\Pi\sigma$ (or of strict germs, following Favre terminology [12]) a birational representative ? Clearly $\Pi\sigma$ is birational if and only if σ is birational.

Known results:

- If S is a Enoki surface (see [7]) or a Inoue-Hirzebruch surface (see [4]) with second Betti number $b_2(S) = n$, known normal forms, namely

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2), \quad 0 < |t| < 1,$$

  and

$$N(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s),$$

  respectively, are birational. Here $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl(2, \mathbb{Z})$ is the composition of n matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

  with at least one of the second type.

- If S is of intermediate type (see definition in section 2), there are normal forms due to C.Favre [12]

$$F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2), z_2^k), \qquad \lambda \in \mathbb{C}^\star,\ \mathfrak{s} \in \mathbb{N}^\star,\ k \geq 2,$$

  where P is a special polynomial. These normal forms are adapted to logarithmic deformations and show the existence of a foliation, however *are not birational*. In [21] K.Oeljeklaus and M.Toma explain how to recover second Betti number n which is now hidden and give coarse moduli spaces of surfaces with fixed intersection matrix,

- Some special cases of intermediate surfaces are obtained from Hénon mappings H or composition of Hénon mappings. More precisely, the germ of H at the fixed point at infinity is strict, hence gives a surface with a GSS [13, 9]. These germs are birational.

Motivation:
Let S be a minimal compact complex surface with Betti numbers $b_1(S) = 1$, $n = b_2(S) > 0$, the class of such surfaces will be denoted VII_0^+. We consider the following conditions:

(A) S contains a global spherical shell (GSS),

(B) S contains $b_2(S)$ rational curves,

(C) S contains a cycle of rational curves,

(D) S admits a deformation into $b_2(S)$ times blown up Hopf surfaces.

GSS Conjecture: *All these properties are equivalent, and any class VII_0^+ surface possesses a global spherical shell (GSS) i.e. an open submanifold biholomorphic to a standard neighborhood of S^3 in $\mathbb{C}^2$ which does not disconnect the surface.*

We have

$$(A) \Longleftrightarrow (B) \Longrightarrow (C) \Longrightarrow (D)$$

In fact $(A) \Longrightarrow (B)$ by the construction of GSS surfaces and $(B) \Longrightarrow (A)$ by [10],
$(A) \Longrightarrow (C)$ also by construction (see [3]) The implication $(C) \Longrightarrow (D)$ has been obtained by I. Nakamura [19, 20].
The strategy developped in [23, 24] is aimed to show that any surface in VII_0^+ satisfies condition (C), therefore the solution to the following problem would be a step toward the conjecture:
Problem: Let $\mathcal{S} \to \Delta$ be a family of compact surfaces over the disc such that for every $u \in \Delta^\star$, S_u contains a GSS. Does S_0 contain a GSS ? In other words, are the surface with GSS closed in

families ?
To solve this problem we have to study families of surfaces in which curves do not belong to flat families, the volume of some curves in these families may be not uniformly bounded (see [11]) and configurations of curves change. Favre normal forms of polynomial germs associated to surfaces with GSS, cannot be used because the discriminant of the intersection form is fixed. Moreover, if using the algorithm of [21] we put F under the form $\Pi\sigma$, σ is not fixed in the logarithmic family, depends on the blown up points and degenerates when a generic blown up point approaches the intersection of two curves.
Therefore this article focuses on the problem of finding new normal forms of contracting germs in intermediate cases of surfaces with fixed simple birational σ, such that surfaces are minimal or not and intersection matrices are not fixed. Since usual holomorphic objects, curves or foliations, do not fit in global family, it turns out that these birational structures depending on a finite number of parameters (in fact the number of parameters is exactly the dimension the local moduli spaces) could be the adapted notion. Moreover, our construction gives a contracting map $G = \Pi\sigma$ unique up to conjugation by elements of a group L of diagonal linear mappings with coefficients equal to roots of unity. The existence of our special birational structure gives rise to developing mappings $\widetilde{Dev}_j : \tilde{S} \to \mathbb{P}^2(\mathbb{C})$ which contract an infinite number of rational curves onto a point P. The inverse image of a small sphere by $\widetilde{Dev}_j$ gives a spherical shell in $\tilde{S}$, hence a GSS in S. This observation will be useful to prove the GSS conjecture.

This article is organized in the following way:
In **section 2**, we introduce general notions on G-structures when $G = Bir(\mathbb{P}^2(\mathbb{C}))$, developing meromorphic mappings and recall known results on affine and projective structures, these being particular cases of birational structures.
In **section 3**, we recall, in order to be self-contained, basic facts on surfaces with global spherical shells (GSS) which will be used all along this article, large families of marked surfaces with GSS which have been introduced in [6]. The proof of the main results hinge upon the fact that in all conjugation class of contracting germs of the form $\Pi\sigma$ there is a Favre germ. In order to be complete and to have clear notations we recall results on Oeljeklaus-Toma logarithmic moduli spaces of Kato surfaces with fixed intersection matrix of rational curves [21] with a slight modification (see Remark 3.18). Favre germs F which correspond to Kato surfaces which have a cycle with ρ branches split into ρ polynomial germs of simple type $F = F_1 \circ \cdots \circ F_\rho$. Let $\mathcal{F}(\mathfrak{s}, k, j)$ be the family of Favre germs of simple type (see Def. 3.17),

$$F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + \sum_{i=j}^{\mathfrak{s}} b_i z_2^i + c z_2^{\frac{\mathfrak{s}k}{k-1}},\ z_2^k)$$

The associated surfaces have a cycle with exactly one branch.
In **section 4** new germs are defined, obtained by composition of n blowing-ups ($2n$ parameters) and, if global twisted vector fields are expected to exist, of an extra invertible polynomial mapping tangent to the identity (the extra parameter a_{l+K}). If the surface contains a cycle of rational curves with only one branch this class of birational contracting germs is denoted by $\mathcal{G} = \mathcal{G}(p, q, r, s, l)$ and have the following form

- if the first blowing-up is not generic

$$G(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i \left(z_1^r z_2^s\right)^{i+1} + a_{l+K} \left(z_1^r z_2^s\right)^{l+K+1}, z_1^r z_2^s \right),$$

  where $K = \max\left\{0, \left[\frac{l-d}{r+s-1}\right]\right\}$, $a_0 \in \mathbb{C}^\star$, $a_i \in \mathbb{C}$, $i = 1, \ldots, l-1, l+K$, and

- if the first blowing-up is generic

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^p z_2^q,\ \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^r z_2^s \right)$$

Among the blowing-ups there are l generic blowing-ups, and $n-l$ non generic, determined by the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl(2,\mathbb{Z}).$$

First, we establish a correspondance between both families $\mathcal{F}(\mathfrak{s},k,j)$ and $\mathcal{G}=\mathcal{G}(p,q,r,s,l)$ when they correspond to the same sequence of blowing-ups giving the same intersection matrix of the rational curves. We provide precise relations between the integers involved in the construction and we explicit conditions which insure the existence of global vector fields.
We denote by $\Phi=\Phi(p,q,r,s,l)$ the group of the germs of biholomorphisms $\varphi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ for which there exists $G,G'\in\mathcal{G}$ such that $G'=\varphi^{-1}G\varphi\in\mathcal{G}$. Let $L:=L(p,q,r,s,l)$ be the group of diagonal linear mappings $\varphi_{A,B}(z_1,z_2)=(Az_1,Bz_2)$ where A,B satisfy the condition

$$B=A^rB^s,\quad A=A^{p+rl}B^{q+sl}$$

Then the following holds (see Prop. 4.33 for a more detailed statement):

Proposition 1. 1 (unicity) *There is an exact sequence*

$$0\to(\mathbb{C},+)\to\Phi\to L\to Id$$

Moreover if $a_{l+K}=0$, then $\Phi=L$, i.e. the birational germ G is unique up to a conjugation by a diagonal linear mapping whose coefficients are roots of unity.

Moving the parameters we have large families of surfaces with base B_J. Is the canonical image of a stratum $B_{J,M}$ of surfaces with fixed intersection matrix M in the Oeljeklaus-Toma coarse moduli space open ? Do we obtain all possible surfaces ?
We know by [6] that outside the hypersurface $T_{J,\sigma}$ the family is versal. Here we show that $T_{J,\sigma}$ is a ramification hypersurface, in particular the canonical mapping from a stratum $B_{J,M}$ to the Oeljeklaus-Toma coarse moduli space is a ramified covering, the mapping is surjective and vanishing of cohomology classes is due to ramification phenomena at $T_{J,\sigma}\cap B_{J,M}$. More precisely (see section 3.4) we have the existence theorem

Theorem 1. 2 (Main theorem) *Denote $\mathfrak{s}:=p+q+l-1$ and $d:=(r+s)-(p+q)$. We choose $a_0\in\mathbb{C}^\star$ and ϵ such that $\epsilon^{r+s-1}=1$. Then*
A) If $r+s-1$ does not divide $l-d$ or $\lambda\neq 1$ there is a bijective polynomial mapping

$$\begin{array}{cccc} f_{a_0,\epsilon}: & \mathbb{C}^{l-1} & \longrightarrow & \mathbb{C}^{l-1} \\ & a=(a_1,\ldots,a_{l-1}) & \longmapsto & \Big(b_{p+q+1}(a),\ldots,b_{p+q+l-1}(a)\Big) \end{array}$$

such that

$$G(z_1,z_2)=\left(\Big(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}\Big)^pz_2^q,\ \Big(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}\Big)^rz_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1,z_2)=\Big(\lambda z_1z_2^{\mathfrak{s}}+\sum_{i=p+q}^{\mathfrak{s}}b_iz_2^i,\ z_2^{r+s}\Big),$$

where λ depends only on a_0 by 4.35.
B) If $l-d=K(r+s-1)$ and $\lambda=1$, there is a bijective polynomial mapping

$$\begin{array}{cccc} f_{a_0,\epsilon}: & \mathbb{C}^{l-1}\times\mathbb{C} & \longrightarrow & \mathbb{C}^{l-1}\times\mathbb{C} \\ & a=(a_1,\ldots,a_{l-1},a_{l+K}) & \longmapsto & \Big(b_{p+q+1}(a),\ldots,b_{p+q+l-1}(a),c(a)\Big) \end{array}$$

such that

$$G(z_1,z_2)=\left(\Big(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}+a_{l+K}z_2^{2l+K}\Big)^pz_2^q,\ \Big(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}+a_{l+K}z_2^{2l+K}\Big)^rz_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1,z_2)=\Big(\lambda z_1 z_2^{\mathfrak{s}}+\sum_{k=p+q}^{\mathfrak{s}} b_k z_2^k + c z_2^{\frac{\mathfrak{s}k(S)}{k(S)-1}}, z_2^{r+s}\Big).$$

Corollary 1. 3 *Let S in class VII_0^+ containing a GSS. Suppose that the dual graph of the rational curves contains a cycle with only one branch, then S admits a birational structure. In particular Kato surfaces admit birational structures provided that $b_2(S)\leq 3$.*

In there are $\rho>1$ branches, there is for each intersection matrix M an open set in the moduli space of Kato surfaces with intersection matrix M obtained by birational germs $G_1\circ\cdots\circ G_\rho$ (see Cor. 1.3 in [6]).
In last section we show how to recover GSS from the existence of developing mappings $\widetilde{Dev}_j:\tilde S\to\mathbb{P}^2(\mathbb{C})$.

2 Birational structures on complex manifolds

2.1 preliminaries

Here are classical definitions as in [14, 15], in the context of complex manifolds. Notice that $\dim X=\dim Y$.

Definition 2. 4 *Let Y be a complex manifold, G a Lie group acting holomorphically on Y on the left and X a complex manifold of dimension n. A (G,Y)-structure on X is a maximal atlas of X, $\phi_i:U_i\to Y$ such that transition maps*

$$\phi_{ij}:=\phi_i\circ\phi_j^{-1}:\phi_j(U_i\cap U_j)\to\phi_i(U_i\cap U_j)$$

are locally elements of G.
Given two (G,Y)-manifolds, a (G,Y)-morphism $f:X_1\to X_2$ is a holomorphic mapping such that for any charts $\phi_i:U_i\to Y$, $\psi_j:V_j\to Y$ of X_1 and X_2 respectively and every connected component C of $U_i\cap f^{-1}(V_j)$, there exists $g\in G$ such that

$$f_{|C}=\psi_j^{-1}\circ g\circ\phi_i.$$

An affine structure (resp. a projective structure) on X is a (G,Y)-structure where $Y=\mathbb{C}^n$ and G is the affine group $A(n,\mathbb{C})=Gl(n,\mathbb{C})\rtimes\mathbb{C}^n$ (resp. $Y=\mathbb{P}^n(\mathbb{C})$ and $G=\mathbb{P}Gl(n+1,\mathbb{C})$).

If $f:X_1\to X_2$ is a local diffeomorphism and X_2 is a (G,Y)-manifold, there exists a unique (G,Y)-structure on X_1 such that f is a morphism of (G,Y)-manifolds. In particular if f is a non ramified covering, X_1 has a canonical (G,Y)-structure.
Taking now $Y=\mathbb{P}^n(\mathbb{C})$ and $G=Bir(\mathbb{P}^n(\mathbb{C}))$, G is neither an algebraic group nor a finite dimensional Lie group [2]. Therefore we extend the previous definition:

Definition 2. 5 *Let X be a complex manifold of dimension n. We say that X admits a birational structure if there is an atlas $(U_i,\varphi_i)_{i\in I}$ such that holomorphic transition maps $b_{ij}:=\varphi_i\circ\varphi_j^{-1}:\varphi_j(U_i\cap U_j)\to\varphi_i(U_i\cap U_j)$ are the restriction of birational mappings of $\mathbb{P}^n(\mathbb{C})$.*

Affine or projective structures are birational structures. If X admits a birational structure and $\Pi:X'\to X$ is a blowup, then X' admits a unique birational structure such that Π is a $(Bir(\mathbb{P}^n(\mathbb{C}),\mathbb{P}^n(\mathbb{C}))$-morphism.

Example 2. 6 *Let X be compact riemann surface, then X admits a birational structure. In fact, if $g(X)\geq 2$, X is the quotient of the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}\mid\Im z>0\}$ by a Fuchsian group, hence a subgroup of $PSL(2,\mathbb{R})$ which acts on $\mathbb{H}$ as $\begin{pmatrix}a&b\\c&d\end{pmatrix}.z=\frac{az+b}{cz+d}$.*

As when G is a Lie group we have a developing mapping $Dev : \tilde{X} \to \mathbb{P}^n(\mathbb{C})$, however Dev is now meromorphic.

Lemma 2. 7 *Let X be a complex manifold endowed with a birational structure $(U_i, \varphi_i)_{i\in I}$ and $p : \tilde{X} \to X$ its universal covering space. Let $x \in X$, $\Gamma = \pi_1(X, x)$ be the fundamental group with base point x. Then for each $x_0 \in p^{-1}(x)$ there is a Γ-equivariant meromorphic developing mapping $Dev_{x_0} : \tilde{X} \to \mathbb{P}^n(\mathbb{C})$, in other words there is a group morphism $h : \Gamma \to Bir(\mathbb{P}^n(\mathbb{C}))$ such that*

$$\forall \gamma \in \Gamma, \quad Dev_{x_0} \circ \gamma = h(\gamma) \circ Dev_{x_0}.$$

Moreover Dev_{x_0} is holomorphic in a neighbourhood of x_0.

Proof: It is sufficient to prove the extension along any path with base point $x_0 \in p^{-1}(x)$. Let $x_1 \in \tilde{X}$ and $\gamma : [0,1] \to \tilde{X}$ a path joining $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. We cover $\gamma([0,1])$ by open domains of charts $(U_0, \varphi_0), \dots, (U_p, \varphi_p)$, such that $U_i \cap U_j \neq \emptyset$ if and only if $0 \leq i \leq p-1$ and $j = i+1$. We prove by induction on $1 \leq j \leq p$ that $\varphi_0 : U_0 \to \mathbb{P}^n(\mathbb{C})$ admits a meromorphic extension Dev_{x_0} on $U_0 \cup \dots \cup U_j$. For $j = 1$, setting $Dev_{x_0|U_0} = \varphi_0$ and $Dev_{x_0|U_1} = b_{01} \circ \varphi_1$. Let $b_{i,i+1} := \varphi_i \circ \varphi_{i+1}^{-1} : \varphi_{i+1}(U_{i,i+1}) \to \varphi_i(U_{i,i+1})$. By assumption $b_{i,i+1}$ extends birationally to $\mathbb{P}^n(\mathbb{C})$. We suppose that Dev_{x_0} has been extended along $U_0 \cup \dots \cup U_{j-1}$ for $j \geq 1$ setting

$$Dev_{x_0|U_{j-1}} : U_{j-1} \to \mathbb{P}^n(\mathbb{C}), \quad x \mapsto Dev_{x_0}(x) = b_{01} \circ \dots \circ b_{j-2,j-1} \circ \varphi_{j-1}(x)$$

and we define

$$Dev_{x_0|U_j} : U_j \to \mathbb{P}^n(\mathbb{C}), \quad x \mapsto Dev_{x_0}(x) = b_{01} \circ \dots \circ b_{j-1,j} \circ \varphi_j(x)$$

For $x \in U_{j-1} \cap U_j$, we have $Dev_{x_0|U_{j-1}}(x) = Dev_{x_0|U_j}(x)$. □

From the lemma we obtain immediately

Proposition 2. 8 *If X is compact simply connected of dimension n and admits a birational structure then the algebraic dimension of X is equal to n.*

Corollary 2. 9 *A non projective K3 surface has no birational structure.*

2.2 Birational structures on non-Kähler complex surfaces

Theorem 2. 10 ([1, 16, 17]) *Any compact complex non-Kählerian surface has a unique minimal model X in the following classes*

- *Class VI_0, $b_1(X)$ is odd and geometric genus satisfies $p_g > 0$. X is an elliptic surface and admits, by [14], an holomorphic affine structure.*
- *Class VII_0, $b_1(X) = 1$, $p_g = 0$, and Kodaira dimension $\kappa(X) \leq 0$.*
  - *(i) If $\kappa(X) = -\infty$, $b_2(X) = 0$ and X contains a curve, then X is a Hopf surface [16] and admits a finite covering by a primary Hopf surface. Any primary Hopf surface is isomorphic to $\mathbb{C}^2 \setminus \{0\}/H$ where H is an infinite cyclic group generated by a contraction*

    $$g : (z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2), \quad 0 < |\alpha| \leq |\beta| < 1, \quad (\beta^m - \alpha)\lambda = 0, \quad m \geq 1.$$

    If $(m-1)\lambda = 0$, X admits a holomorphic affine structure [18], p93. In all cases the contraction is an invertible polynomial mapping hence birational, therefore X admits a birational structure.
  - *(ii) If $\kappa(X) = -\infty$, $b_2(X) = 0$ and X contains no curve, then X is a Inoue surface by [22] and admits a holomorphic affine structure [14].*
  - *(iii) If $\kappa(X) = 0$, then X is a secondary Kodaira surface [1], $b_2(X) = 0$. By [14], X admits an affine holomorphic structure.*
  - *(iv) Surfaces with $b_2(X) > 0$. The only known surfaces are Kato surfaces.*

All compact non-Kähler surfaces admit affine structures but some Hopf surfaces and surfaces in class VII_0^+.
Conjecture: Any compact non-Kähler complex surface admits a birational structure.

3 Surfaces with Global Spherical Shells

3.1 Basic constructions

Definition 3. 11 *Let S be a compact complex surface. We say that S contains a global spherical shell, if there is a biholomorphic map $\varphi : U \to S$ from a neighbourhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere S^3 into S such that $S \setminus \varphi(S^3)$ is connected.*

Hopf surfaces are the simplest examples of surfaces with GSS.

Let S be a surface containing a GSS with $n = b_2(S)$. It is known that S contains n rational curves and to each curve it is possible to associate a contracting germ of mapping $F = \Pi\sigma = \Pi_0 \cdots \Pi_{n-1}\sigma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ where $\Pi = \Pi_0 \cdots \Pi_{n-1} : B^\Pi \to B$ is a sequence of n blowing-ups and σ is a germ of isomorphism (see [3]).

Definition 3. 12 *Let S be a surface containing a GSS, with $n = b_2(S)$. A* **Enoki covering** *of S is an open covering $\mathcal{U} = (U_i)_{0\le i\le n-1}$ obtained in the following way:*

- *W_0 is the ball of radius $1+\epsilon$ blown up at the origin, $C_0 = \Pi_0^{-1}(0)$, $B'_0 \subset\subset B_0$ are small balls centered at $O_0 = (a_0, 0) \in W_0$, $U_0 = W_0 \setminus B'_0$,*
- *For $1 \le i \le n-1$, W_i is the ball B_{i-1} blown up at O_{i-1}, $C_i = \Pi_i^{-1}(O_{i-1})$, $B'_i \subset\subset B_i$ are small balls centered at $O_i \in W_i$, $U_i = W_i \setminus B'_i$.*

The pseudoconcave boundary of U_i is patched with the pseudoconvex boundary of U_{i+1} by Π_i, for $i = 0, \ldots, n-2$ and the pseudoconcave boundary of U_{n-1} is patched with the pseudoconvex boundary of U_0 by $\sigma\Pi_0$, where

$$\begin{array}{rccc} \sigma : & B(1+\epsilon) & \to & W_{n-1} \\ & z = (z_1, z_2) & \mapsto & \sigma(z) \end{array}$$

is biholomorphic on its image, satisfying $\sigma(0) = O_{n-1}$.

If we want to obtain a minimal surface, the sequence of blowing-ups has to be made in the following way:

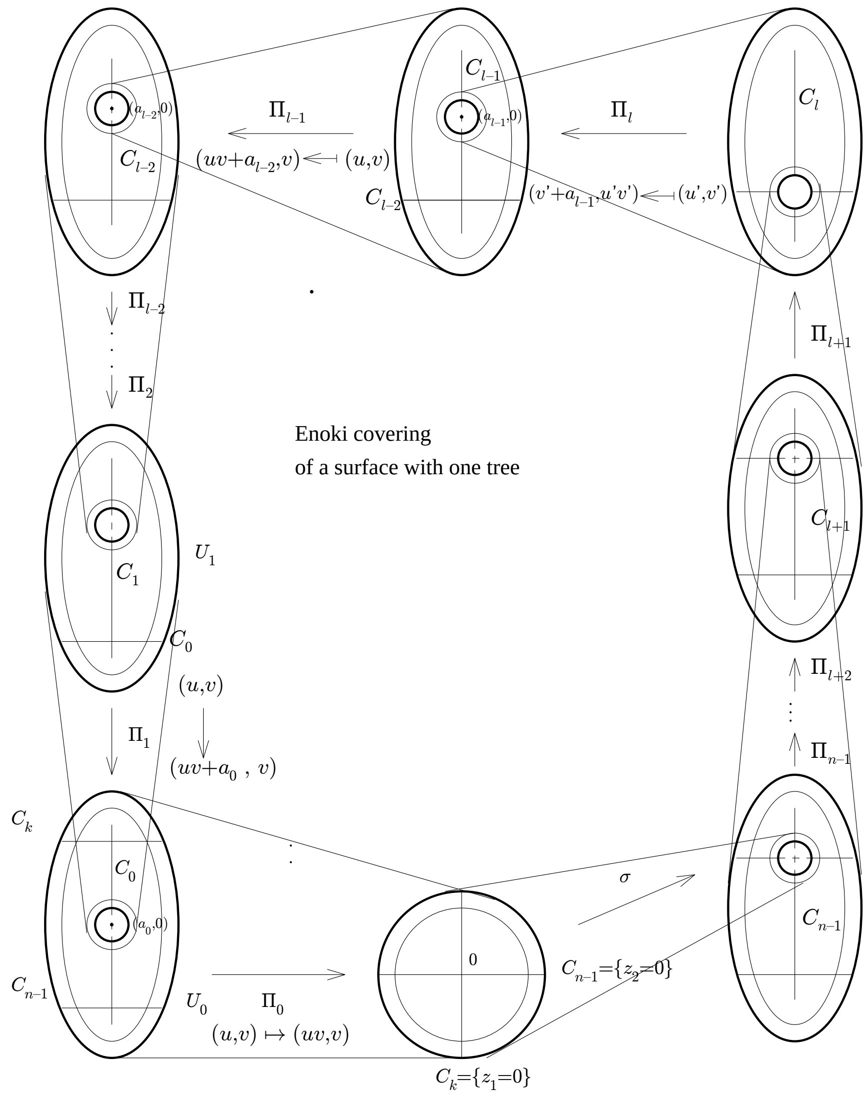
- Π_0 blows up the origin of the two dimensional unit ball B,
- Π_1 blows up a point $O_0 \in C_0 = \Pi_0^{-1}(0)$,. . .
- Π_{i+1} blows up a point $O_i \in C_i = \Pi_i^{-1}(O_{i-1})$, for $i = 0, \ldots, n-2$, and
- $\sigma : \bar{B} \to B^\Pi$ sends isomorphically a neighbourhood of $\bar{B}$ onto a small ball in B^Π in such a way that $\sigma(0) \in C_{n-1}$.

Each W_i is covered by two charts with coordinates (u_i, v_i) and (u'_i, v'_i) in which Π_i writes $\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i)$ and $\Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i)$. In these charts the exceptional curves has always the equations $v_i = 0$ and $v'_i = 0$.
A blown up point $O_i \in C_i$ will be called **generic** if it is not at the intersection of two curves. The data (S, C) of a surface S and of a rational curve in S will be called a **marked surface**.

We assume that S is minimal and that we are in the intermediate case, therefore there is at least one blowing-up at a generic point, and one at the intersection of two curves (hence $n \ge 2$). If there is only one branch i.e. one regular sequence and if we choose C_0 as being the curve which induces the root of the branch, we suppose that

- Π_1 is a generic blowing-up,
- Π_{n-1} blows-up the intersection of C_{n-2} with another rational curve and
- $\sigma(0)$ is one of the two intersection points of C_{n-1} with the previous curves.

The Enoki covering is obtained as in the following picture:



where

- $1 \le l \le n-1$ and $n \ge 2$. If all, but one, blowing-ups are generic, then $l = n-1$
- For $i = 1, \ldots, l-1$, $\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i)$ are generic blowing-ups,
- $\Pi_l(u'_l, v'_l) = (v'_l + a_{l-1}, u'_l v'_l)$ is also generic, but O_l is the origin of the chart (u'_l, v'_l),
- For $i = l+1, \ldots, n-1$, $\Pi_i(u_i, v_i) = (u_i v_i, v_i)$ or $\Pi_i(u'_i, v'_i) = (v'_i, u'_i v'_i)$ are blowing-ups at the intersection of two curves.

3.2 Large families of marked surfaces

With the previous notations, we consider global families of minimal compact surfaces with the same charts, parameterized by the coordinates of the blown up points on the successive exceptional curves obtained in the construction of the surfaces and such that any marked surface with GSS (S, C_0) belongs to at least one of these families. More precisely, let $F(z) = \Pi_0 \cdots \Pi_{n-1}\sigma(z)$ be a germ associated to any marked surface (S, C_0) with $tr(S) = 0$. In order to fix the notations we suppose that $C_0 = \Pi_0^{-1}(0)$ meets two other curves (see the picture after definition 3.12), hence $\sigma(0)$ is the intersection of C_{n-1} with another curve. We suppose that

$$\partial_1 \sigma_2(0) = 0.$$

We denote by $I_\infty(C_0) \subset \{0, \dots, n-1\}$ the subset of indices which correspond to blown up points at infinity, that is to say,

$$I_\infty(C_0) := \big\{ i \mid O_i \text{ is the origin of the chart } (u'_i, v'_i) \big\}.$$

Each generic blow-up

$$\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i) \quad \text{or} \quad \Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i)$$

may be deformed moving the blown up point $(a_{i-1}, 0)$. If we do not want to change the configuration we take

for all $\kappa = 0, \dots, \rho - 1$ (with $n_0 = 0$),

$$\begin{cases} a_{n_1+\cdots+n_\kappa} \in \mathbb{C}^\star, \\ \forall i,\ 1 \le i \le l_\kappa - 1, \qquad a_{n_1+\cdots+n_\kappa+i} \in \mathbb{C}, \\ \forall j,\ 0 \le j \le n_{\kappa+1} - l_\kappa - 1, \qquad a_{n_1+\cdots+n_\kappa+l_\kappa+j} = 0. \end{cases}$$

The mapping σ is supposed to be fixed. We obtain a large family of compact surfaces which contains S such that all the surfaces S_a have the same intersection matrix

$$M = M(S_a) = M(S),$$

therefore are logarithmic deformations. For $J = I_\infty(C_0)$ we denote this family

$$\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \to B_{J,M}$$

where

$$\begin{aligned} & B_{J,M} \\ & := \mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \{0\}^{n_1-l_0} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1} \times \{0\}^{n_{\kappa+1}-l_\kappa} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1} \times \{0\}^{n_\rho-l_{\rho-1}} \\ & \simeq \mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1} \end{aligned}$$

and $n_1 + \cdots + n_\rho = n$.

In $\mathcal{S}_{J,M,\sigma}$ there is a flat family of divisors $\mathcal{D} \subset \mathcal{S}$ with irreducible components

$$\mathcal{D}_i, \quad i = 0, \dots, n-1,$$

such that for every $a \in B_{J,M}$, $M = (D_{i,a}.D_{j,a})_{0 \le i,j \le n-1}$. We may extend this family towards smaller or larger strata which produce minimal surfaces:

- On one hand, **towards a unique Inoue-Hirzebruch surface**: Over

  $$\mathbb{C}^{l_0} \times \{0\}^{n_1-l_0} \times \cdots \times \mathbb{C}^{l_\kappa} \times \{0\}^{n_{\kappa+1}-l_\kappa} \times \cdots \times \mathbb{C}^{l_{\rho-1}} \times \{0\}^{n_\rho-l_{\rho-1}} \simeq \mathbb{C}^{l_0} \times \cdots \times \mathbb{C}^{l_\kappa} \times \cdots \times \mathbb{C}^{l_{\rho-1}},$$

  $$\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \to \mathbb{C}^{l_0} \times \cdots \times \mathbb{C}^{l_\kappa} \times \cdots \times \mathbb{C}^{l_{\rho-1}}.$$

  If for an index κ, $a_{n_1+\cdots+n_\kappa} = 0$, there is a jump in the configuration of the curves. For instance, if for all κ, $\kappa = 0, \dots, \rho - 1$

  $$a_{n_1+\cdots+n_\kappa} = \cdots = a_{n_1+\cdots+n_\kappa+l_\kappa-1} = 0$$

  we obtain a Inoue-Hirzebruch surface. To be more precise the base

  $$\mathbb{C}^{l_0} \times \cdots \times \mathbb{C}^{l_\kappa} \times \cdots \times \mathbb{C}^{l_{\rho-1}}$$

  splits into locally closed submanifolds called **strata**

- the Zariski open set $\mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa -1} \times \cdots \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}$,
- $\rho = C^1_\rho$ codimension one strata

$$\mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \cdots \times \{0\} \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa -2} \times \cdots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}, \quad 0 \le \kappa \le \rho - 1,$$

- $C^p_{\rho+p-1}$ codimension p strata, $1 \le p := p_0 + \cdots + p_{\rho-1} \le l_0 + \cdots + l_{\rho-1}$,

$$\{0\}^{p_0} \times \mathbb{C}^\star \times \mathbb{C}^{l_0-p_0-1} \times \cdots \times \{0\}^{p_\kappa} \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa - p_\kappa -1} \times \cdots \times \{0\}^{p_{\rho-1}} \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-p_{\rho-1}-1}$$

- On second hand, **towards Enoki surfaces**. If for all indices such that O_i is at the intersection of two rational curves, in particular for $i \in J$, the blown up point O_i is moved to $O_i = (a_i, 0)$ with $a_i \neq 0$, all the blown up points become generic, the trace of the contracting germ is different from 0. We obtain also all the intermediate configurations.

Proposition 3. 13 ([6] Prop.2.6) *There is a monomial holomorphic function $t : \mathbb{C}^{\mathrm{Card}\, J} \to \mathbb{C}$ depending on the variables a_j, $j \in J$ such that over $B_J := \{|t(a)| < 1\} \subset \mathbb{C}^n$, the family $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \to B_J$ may be extended and for every $a \in B_J$, $t(a) = \mathrm{tr}\,(S_a)$.*

Remain non minimal surfaces: we still extend the previous family on a small neighbourhood $\widehat{B_J}$ of B_J, moving the blown up point transversally to the exceptional curves $C_i = \{v_i = 0\} \cup \{v'_i = 0\}$, introducing n new parameters

$$\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i + b_{i-1}), \quad \text{or} \quad \Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i + b_{i-1}), \quad |b_{i-1}| << 1,$$

we obtain

$$\widehat{\Phi}_{J,\sigma} : \widehat{\mathcal{S}}_{J,\sigma} \to \widehat{B}_J,$$

with $\dim \widehat{B_J} = 2n = 2b_2$. Since for any $(a,b) \in \widehat{B_J}$, $h^1(S_{a,b}, \Theta_{a,b}) = 2b_2(S_{a,b}) + h^0(S_{a,b}, \Theta_{a,b})$, there are some questions:

- Are the parameters a_i, b_i, $i = 0, \ldots, n-1$, effective ? By [6], they are generically effective.
- Which parameter to add when $h^1(S_{a,b}, \Theta_{a,b}) = 2b_2(S_{a,b}) + 1$ in order to obtain a complete family ?
- If we choose $\sigma = Id$ or more generally an invertible polynomial mapping, we obtain a birational polynomial germs. Does this families contain all the isomorphy classes of surfaces with fixed intersection matrix M ?

Remark 3. 14 *It is difficult to determine the maximal domain $\widehat{B}_J$ over which $\widehat{\Phi}_{J,\sigma}$ may be defined. When the surface is minimal, i.e. when $b = (b_0, \ldots, b_{n-1}) = 0$, $F_{a,b}(0) = 0$. However, when $b \neq 0$, the fixed point $\zeta = (\zeta_1, \zeta_2)$ moves and the existence condition for the corresponding surface is that the eigenvalues λ_1 and λ_2 of $DF_{a,b}(\zeta)$ satisfy $|\lambda_i| < 1$, $i = 1, 2$.*

3.3 Oeljeklaus-Toma logarithmically versal family

The goal is to compare the Oeljeklaus-Toma logarithmic families of surfaces with the strata in large families of surfaces $\Phi_{J,M\sigma} : \mathcal{S}_{J,M,\sigma} \to B_{J,M}$ which have the same intersection matrix M. In the case of surfaces with only one branch it turns out that we obtain all the surfaces.
We recall the results of [21] used in the sequel with a small correction described in the remark 3.18. All surfaces of intermediate type may be obtained from a polynomial germ in the following normal form obtained by [12] and improved by [21].

$$(CG) \qquad F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, z_2^k)$$

where $k, \mathfrak{s} \in \mathbb{Z}$, $k > 1$, $\mathfrak{s} > 0$, $\lambda \in \mathbb{C}^\star$,

$$P(z_2) = c_j z_2^j + c_{j+1} z_2^{j+1} + \cdots + c_{\mathfrak{s}} z_2^{\mathfrak{s}}$$

is a complex polynomial satisfying the conditions

$$0 < j < k, \quad j \leq \mathfrak{s}, \quad c_j = 1, \quad c \in \mathbb{C}, \quad \gcd\{k, m \mid c_m \neq 0\} = 1$$

with $c = 0$ whenever $\frac{\mathfrak{s}k}{k-1} \notin \mathbb{Z}$ or $\lambda \neq 1$.

Lemma 3. 15 ([21],§4) *Two polynomial germs F and*

$$\tilde{F}(z_1, z_2) = \left(\tilde{\lambda} z_1 z_2^{\tilde{\mathfrak{s}}} + \tilde{P}(z_2) + \tilde{c} z_2^{\frac{\tilde{\mathfrak{s}}\tilde{k}}{\tilde{k}-1}},\ z_2^{\tilde{k}}\right),$$

in normal form (CG) are conjugated if and only if there exists $\epsilon \in \mathbb{C}$, $\epsilon^{k-1} = 1$ such that

$$\tilde{k} = k, \quad \tilde{\mathfrak{s}} = \mathfrak{s}, \quad \tilde{\lambda} = \epsilon^{\mathfrak{s}} \lambda, \quad \tilde{P}(z_2) = \epsilon^{-j} P(\epsilon z_2), \quad \tilde{c} = \epsilon^{\frac{\mathfrak{s}k}{k-1}} c.$$

Intermediate surfaces admitting a global non-trivial twisted vector field or a non-trivial section of the anticanonical line bundle are exactly those for which $(k-1) \mid \mathfrak{s}$. When moreover $\lambda = 1$ we have a non-trivial global vector field.

Definition 3. 16 *Let S be a surface containing a GSS. The least integer $\mu \geq 1$ such that there exists $\kappa \in \mathbb{C}^\star$ for which*

$$H^0(S, K_S^{-\mu} \otimes L^\kappa) \neq 0$$

is called the **index** *of S.*

If S is defined by the polynomial germ

$$(CG) \qquad F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, z_2^k)$$

then by [21] Remark 4.5,

$$index(S) := \mu = \frac{k-1}{gcd(k-1, \mathfrak{s})}.$$

Notice that these germs show the existence of a foliation whose leaves are defined by $\{z_2 = constant\}$, however *they are not birational.*

The set of polynomial germs

$$F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2),\ z_2^k)$$

with $c = 0$ are called in pure normal form.

Definition 3. 17 ([21] Def 4.7) *For fixed k and $\mathfrak{s}$ and for a polynomial germ*

$$(CG) \qquad F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, z_2^k)$$

we define inductively the following finite sequences of integers

$$j =: m_1 < \cdots < m_\rho \leq \mathfrak{s}, \quad \text{and} \quad k > i_1 > i_2 > \cdots > i_\rho = 1,$$

by:

(i) $m_1 := j$, $i_1 := \gcd(k, m_1)$,

(ii) $m_\alpha := \min\{m > m_{\alpha-1} \mid c_m \neq 0, \gcd(i_{\alpha-1}, m) < i_{\alpha-1}\}$, $i_\alpha = \gcd(k, m_1, \ldots, m_\alpha) = \gcd(i_{\alpha-1}, m_\alpha)$,

(iii) $1 = i_\rho := \gcd(k, m_1, \ldots, m_{\rho-1}, m_\rho) < \gcd(k, m_1, \ldots, m_{\rho-1})$.

We call $(m_1, \ldots, m_\rho)$ the **type** *of F and ρ the* **length of the type**. *If $\rho = 1$, we say that F is of* **simple type**.

By [21], §6, the length of the type is exactly the number ρ of branches previously introduced.

Remark 3. 18 *1) If the length is $\rho = 1$, then $\gcd(k, j) = 1$ and there is no extra condition on the coefficients $c_{j+1}, \dots, c_{\mathfrak{s}}$, therefore the parameter space of polynomial germs in pure form with integers $k, \mathfrak{s}$ and type j is*

$$U_{k,\mathfrak{s},j} = \mathbb{C}^\star \times \mathbb{C}^{\mathfrak{s}-j}.$$

If the length of the type is $\rho \geq 2$, notice that by definition, we have $c_{m_\alpha} \in \mathbb{C}^\star$, $\alpha = 1, \dots, \rho$, and $c_{m_1} = c_j = 1$, however between c_{m_α} and $c_{m_{\alpha+1}}$, the coefficients

$$c_{m_\alpha + i_\alpha}, c_{m_\alpha + 2i_\alpha}, \dots, c_{m_\alpha + \left[\frac{m_{\alpha+1} - m_\alpha}{i_\alpha}\right] i_\alpha} \in \mathbb{C}$$

may take any value, but all the other coefficients from $c_{m_\alpha+1}$ to $c_{m_{\alpha+1}-1}$ should vanish. Let

$$\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho) := \sum_{\alpha=1}^{\rho-1} \left[\frac{m_{\alpha+1} - m_\alpha}{i_\alpha}\right] + t - m_\rho$$

then the parameter space of all the germs F with the same integers $\mathfrak{s}, k$ and of the same type $(m_1, \dots, m_\rho)$ in pure form are parameterized by

$$(\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}.$$

There exists a family of surfaces

$$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}$$

such that for every $u \in (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}$, S_u is associated to the germ F_u. We have

Theorem and Definition 3. 19 ([21], thm 7.13) *With the above notations we have:*

- *If $k - 1$ does not divide $\mathfrak{s}$, the family*

  $$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\mathfrak{s}} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} =: U_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

  is logarithmically versal at every point and contains all surfaces with parameters $\mathfrak{s}, k$ and type $(m_1, \dots, m_\rho)$.
- *If $k - 1$ divides $\mathfrak{s}$, the family*

  $$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \mathbb{C} =: U_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

  - *is logarithmically complete at every point,*
  - *is logarithmically versal at every point of*

    $$U^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho} := (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \mathbb{C}$$

    and its restriction

    $$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to U^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

    contains all surfaces with parameters $\mathfrak{s}, k$ and type $(m_1, \dots, m_\rho)$ admitting a non-trivial global vector field,

  Moreover its restrition

  $$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to \mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} := U^{\lambda\neq 1, c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

  is logarithmically versal at every point and contains all surfaces with parameters k, $\mathfrak{s}$ and type $(m_1, \dots, m_\rho)$ without non-trivial global vector fields.

We shall call this family the **Oeljeklaus-Toma logarithmic family of parameters k, $\mathfrak{s}$ and type** *$(m_1, \dots, m_\rho)$.*

By lemma 315, for fixed $k, \mathfrak{s}$ and type $(m_1, \dots, m_\rho)$, $\mathbb{Z}/(k-1)$ acts on the germs in pure normal form. By [21] (7.14),

- $\mathcal{M}_{k,\mathfrak{s},m_1,\dots,m_\rho} := U_{k,\mathfrak{s},m_1,\dots,m_\rho}/\big(\mathbb{Z}/(k-1)\big) \qquad$ if $k-1$ does not divide $\mathfrak{s}$,

- $\begin{cases} \mathcal{M}^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho} := U^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}/\big(\mathbb{Z}/(k-1)\big) \\ \\ \mathcal{M}^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho} := U^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho}/\big(\mathbb{Z}/(k-1)\big), \end{cases}$ if $k-1$ divides $\mathfrak{s}$,

are coarse moduli spaces, moreover the canonical mappings are ramified covering spaces. By lemma 3.15, the ramification set is the union $T_{k,\mathfrak{s},m_1,\dots,m_\rho}$ (resp. $T^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}$, $T^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho}$) of hypersurfaces $\{c_i = 0\}$, with $j+1 \le i \le \mathfrak{s}$ such that $c_i \in \mathbb{C}$, in particular

$$U_{k,\mathfrak{s},m_1,\dots,m_\rho} \setminus T_{k,\mathfrak{s},m_1,\dots,m_\rho} \to \mathcal{M}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

$$U^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho} \setminus T^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to \mathcal{M}^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

$$U^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho} \setminus T^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to \mathcal{M}^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

are non ramified covering spaces having $k-1$ sheets.

Remark 3. 20 *When $k-1$ divides $\mathfrak{s}$, all the surfaces over the fiber $(\lambda, a, b) \times \mathbb{C}$ with $(\lambda, a, b) \in \mathbb{C} \setminus \{0,1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}$ are isomorphic. Moreover*

$$U^{\lambda=1}_{k,\mathfrak{s},m_1,\dots,m_\rho}/\big(\mathbb{Z}/(k-1)\big) \cup U^{\lambda\neq 1,c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}/\big(\mathbb{Z}/(k-1)\big)$$

is not separated. In fact, denote by

$$F_{\lambda,c}(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, z_2^k).$$

Then any neighbourhood of $F_{1,c}$ with $c \neq 0$ meets any neighbourhood of $F_{1,0}$ because if $\lambda \neq 1$,

$$F_{\lambda,c} \sim F_{\lambda,0}.$$

Proposition and Definition 3. 21 *If $k-1$ divides $\mathfrak{s}$, the restriction*

$$\mathcal{S}^0_{k,\mathfrak{s},m_1,\dots,m_\rho} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} := U^{c=0}_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

of the family

$$\mathcal{S}_{k,\mathfrak{s},m_1,\dots,m_\rho} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \mathbb{C} := U_{k,\mathfrak{s},m_1,\dots,m_\rho}$$

will be called the **Oeljeklaus-Toma family of pure surfaces***. It is versal at every point of*

$$\mathbb{C} \setminus \{0,1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}$$

and effective at every point of

$$\{1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)}.$$

Since the hypersurface $(\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \{0\}$ is invariant under the action of $\mathbb{Z}/(k-1)$ by (15), the projection

$$pr : (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \mathbb{C} \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \{0\}$$

induces a holomorphic mapping

$$p : (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \mathbb{C}/\big(\mathbb{Z}/(k-1)\big) \to (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\mathfrak{s},m_1,\dots,m_\rho)} \times \{0\}/\big(\mathbb{Z}/(k-1)\big).$$

4 Special birational structures on compact surfaces and birational germs

4.1 Birational germs associated to marked surfaces with one branch

4.1.1 Invariants and geometric properties

In this section we define new normal forms of contracting germs, then we determine geometric properties and conditions for the existence of global vector fields.

Let (S, C_0) be a marked surface with GSS and let M be the intersection matrix of the rational curves. We suppose that C_0 is the root of the unique branch (see picture in section 3.1). Then we have

$$\Pi_l \cdots \Pi_{n-1}(u'', v'') = (u''^p v''^q + a_{l-1}, u''^r v''^s)$$

where $(u'', v'') = (u, v)$ or $(u'', v'') = (u', v')$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is the composition of matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $A' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, the last one being equal to A'. We set

$$\delta := ps - qr = \pm 1,$$

$$1 \leq d := (r+s) - (p+q) < r+s.$$

Moreover

$$\Pi_0 \cdots \Pi_{l-1}(u, v) = \left(uv^l + \sum_{i=0}^{l-2} a_i v^{i+1}, v \right)$$

Hence

$$G(z) = \Pi\sigma(z) = \left(\sigma_1(z)^{p+rl} \sigma_2(z)^{q+sl} + \sum_{i=0}^{l-1} a_i \Big(\sigma_1(z)^r \sigma_2(z)^s \Big)^{i+1}, \sigma_1(z)^r \sigma_2(z)^s \right),$$

where σ is a germ of biholomorphism.

If there is no global vector fields the number of parameters given by the blown up points is $2n$ as the expected number of parameters of the versal deformation, therefore the question arises to know if with $\sigma = Id$ we obtain locally versal families. If there are non trivial global vector fields we need (at least) an extra parameter. We add this parameter by the composition $\bar\sigma \Pi_0 \cdots \Pi_{l-1} \Pi_l \cdots \Pi_{n-1} Id$ where

$$\bar\sigma(u, v) = (u + a_{l+K} v^{l+K+1}, v), \quad K \geq 0,$$

where K will be chosen in proposition 23. We obtain a new mapping (denoted in the same way)

$$\begin{aligned} G(z) &= \bar\sigma \Pi_0 \cdots \Pi_{l-1} \Pi_l \cdots \Pi_{n-1} Id(z) \\ &= \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i \big(z_1^r z_2^s\big)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right). \end{aligned}$$

We obtain large families $\mathcal{S}_{J, \sigma_{a_{l+K}} \to B_J}$ and we shall prove that the stratum $B_{J,M}$ is a ramified covering over the OT moduli space of marked surfaces with GSS and intersection matrice M.

Lemma 4. 22 *Let*

$$G(z) = \Pi\sigma(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i \big(z_1^r z_2^s\big)^{i+1} + a_{l+K} \big(z_1^r z_2^s\big)^{l+K+1}, z_1^r z_2^s \right),$$

then the associated surface $S = S(G)$ admits a non trivial global twisted vector field if and only if

$$u = \frac{p+s+rl-1-\delta}{r+s-1}, \quad v = \frac{r+q+sl-1+\delta}{r+s-1}, \quad \text{where} \quad \delta := ps - qr$$

are positive integers. Moreover this twisted vector field is a global vector field if and only if

$$\delta a_0^u k(S) = 1.$$

Proof: We have by a straightforwad computation

$$\det DG(z) = (ps-qr) z_1^{p+r(l+1)-1} z_2^{q+s(l+1)-1}.$$

By [8], there exists a non trivial global twisted vector field $\theta \in H^0(S, \Theta \otimes L^\lambda)$ on S if and only if there is a global twisted section of the anticanonical bundle $\omega \in H^0(S, K^{-1} \otimes L^\kappa)$. Moreover the twisting factors satisfy the relation $\lambda = k(S)\kappa$. The section θ is a global vector field if $\lambda = 1$ i.e.

$$\kappa = \frac{1}{k(S)} \tag{1}$$

Such a section exists if and only if there is a germ of 2-vector field (denoted in the same way)

$$\omega(z) = z_1^u z_2^v A(z) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$$

where $A(0) \neq 0$ such that $\omega(G(z)) = \kappa \det DG(z) \omega(z)$, or equivalently,

$$(a_0 z_1^r z_2^s + \cdots)^u (z_1^r z_2^s)^v A(G(z)) = \kappa (ps-qr) z_1^{p+r(l+1)-1+u} z_2^{q+s(l+1)-1+v} A(z).$$

Comparing terms of lower degree, we obtain the necessary condition

$$a_0^u (z_1^r z_2^s)^{u+v} = \kappa (ps-qr) z_1^{p+r(l+1)-1+u} z_2^{q+s(l+1)-1+v}$$

therefore u and v satisfy the linear system

$$\begin{cases} r(u+v) &= p + r(l+1) - 1 + u \\ s(u+v) &= q + s(l+1) - 1 + v \end{cases}$$

The determinant of the system is $\Delta = -r - s + 1 < 0$ and the solution is

$$u = \frac{p+s+rl-1-\delta}{r+s-1}, \quad v = \frac{r+q+sl-1+\delta}{r+s-1}, \quad \text{where} \quad \delta := ps - qr = \pm 1.$$

Since u and v are the vanishing orders of ω along the curves, a necessary condition for the existence of ω is that u and v are positive integers. Cancelling the common factors we obtain

$$a_0^u = \kappa \delta$$

and with relation (1)

$$\kappa = \delta a_0^u = \frac{1}{k(S)}.$$

If u and v are integers,

$$(a_0 + \cdots)^u A(G(z)) = \kappa \delta A(z),$$

with $a_0 \neq 0$. Setting

$$1 + f(z) = \frac{\kappa \delta}{(a_0 + \cdots)^u},$$

we have

$$A(G(z)) = (1 + f(z)) A(z)$$

Therefore

$$A(z) = \frac{A(0)}{\prod_{j=0}^{\infty}\left(1+f(G^j(z))\right)},$$

the infinite product converges because G is contractant. This proves the existence of ω. $\square$

Proposition 4. 23 *Let*

$$G(z) = \Pi\sigma(z) = \left(z_1^{p+rl}z_2^{q+sl} + \sum_{i=0}^{l-1} a_i\left(z_1^r z_2^s\right)^{i+1} + a_{l+K}\left(z_1^r z_2^s\right)^{l+K+1}, z_1^r z_2^s\right),$$

and let $S = S(G)$ be the associated surface. Then the surface $S(G)$ admits a non trivial global twisted vector field if and only if there exists an integer $k \geq 0$ such that

$$l = d + k(r+s-1),$$

If this condition is fulfilled, we choose

$$K = k$$

and $S(G)$ admits a non trivial vector field if and only if for $u = \frac{p+s+rl-1-\delta}{r+s-1} \in \mathbb{N}^\star$,

$$\delta a_0^u k(S) = 1.$$

Proof: With notations of lemma 4.22, we have to show that u and v are integers if and only if $l = d + k(r+s-1)$.
If u and v are integers,

$$u + v = l + 1 + \frac{p+q+l-1}{r+s-1} \in \mathbb{N},$$

where $p + q < r + s$. Therefore, $l = d + k(r+s-1)$. Conversely, if $l = d + k(r+s-1)$, it is easy to check that u and v are positive integers and the proof is left to the reader. $\square$

Proposition 4. 24 *Let*

$$G(z) = \Pi\sigma(z) = \left(z_1^{p+rl}z_2^{q+sl} + \sum_{i=0}^{l-1} a_i\left(z_1^r z_2^s\right)^{i+1} + a_{l+K}\left(z_1^r z_2^s\right)^{l+K+1}, z_1^r z_2^s\right),$$

and $S = S(G)$ the associated surface. Then

$$k(S) = r + s.$$

Proof: The dual graph of the curves is composed of a cycle with (here) only one chain of rational curves called the tree or the branch. The proof is achieved by induction on the number $N \geq 1$ of singular sequences. We denote as in [3]

$$a(S) = (s_{k_1} \cdots s_{k_N} r_l),$$

where for any $k \geq 1$, s_k is the singular k-sequence $s_k = (k+2, 2, \ldots, 2)$ and r_l is the regular l-sequence $r_l = (2, \ldots, 2)$. We have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_N \end{pmatrix}$$

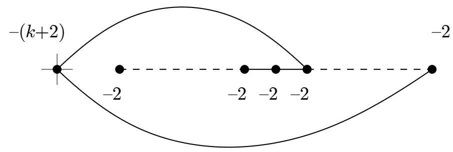
and for any $1 \leq i \leq N$ we set

$$\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} = \begin{pmatrix} p_i(k_1,\ldots,k_i) & q_i(k_1,\ldots,k_i) \\ r_i(k_1,\ldots,k_i) & s_i(k_1,\ldots,k_i) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_i \end{pmatrix},$$

therefore

$$\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} = \begin{pmatrix} q_{i-1} & p_{i-1} + k_i q_{i-1} \\ s_{i-1} & r_{i-1} + k_i s_{i-1} \end{pmatrix} \tag{2}$$

If $N = 1$, dual graph of the curves is



the (opposite) intersection matrix of the (unique) branch is the matrix of a chain of length k

$$\delta_k = \begin{vmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{vmatrix}$$

We have $\delta_k = k+1$ and by [5] thm 3.20, $k(S)$ is equal to δ_k. Now here

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$$

therefore the result is checked for $N=1$.
If $N=2$, the sequence of opposite self-intersections of the curves in the branch is

$$\underbrace{2\cdots 2}_{k_1-1}\,(k_2+2)$$

On one hand

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & k_2 \end{pmatrix} = \begin{pmatrix} 1 & k_2 \\ k_1 & 1+k_1k_2 \end{pmatrix}$$

On second hand, the order of the (opposite) intersection matrix of the branch is k_1. By [5] thm 3.20,

$$k(S) = \begin{vmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & k_2+2 \end{vmatrix} = k_1k_2 + k_1 + 1 = r + s.$$

- If $N = 2\nu$, the sequence of opposite self-intersections of the curves in the branch is

$$\underbrace{2\cdots 2}_{k_1-1}\,(k_2+2)\;\underbrace{2\cdots 2}_{k_3-1}\;\cdots\cdots\cdots\;\underbrace{2\cdots 2}_{k_{2\nu-1}-1}\;(k_{2\nu}+2)$$

- If $N = 2\nu+1$, the sequence of opposite self-intersections is

$$\underbrace{2\cdots 2}_{k_1-1}\,(k_2+2)\;\underbrace{2\cdots 2}_{k_3-1}\;\cdots\cdots\cdots\;\underbrace{2\cdots 2}_{k_{2\nu-1}-1}\;(k_{2\nu}+2)\;\underbrace{2\cdots 22}_{k_{2\nu+1}}$$

- If $N = 2\nu$, we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & k_{2\nu} \end{pmatrix}$$

  the determinant of the opposite self-intersection matrix of the branch is

$$\delta(k_1,\dots,k_{2\nu}) = \begin{vmatrix} \boxed{D} & -1 \\ -1 & k_{2\nu}+2 \end{vmatrix}$$

where $D = D(k_1, \dots, k_{2\nu-1})$ is the block corresponding to

$$\underbrace{2\cdots2}_{k_1-1}\ (k_2+2)\ \underbrace{2\cdots2}_{k_3-1}\ \cdots\cdots\cdots\ \underbrace{2\cdots2}_{k_{2\nu-1}-1}$$

We have by [5], the induction hypothesis and relations (2),

$$\begin{aligned}
k(S) &= \delta(k_1,\dots,k_{2\nu}) = k_{2\nu} \det D(k_1,\dots,k_{2\nu-1}) + \det D(k_1,\dots,k_{2\nu-1}+1) \\
&= k_{2\nu}\Big(r(k_1,\dots,k_{2\nu-2},k_{2\nu-1}-1) + s(k_1,\dots,k_{2\nu-2},k_{2\nu-1}-1)\Big) \\
&\quad + r(k_1,\dots,k_{2\nu-1}) + s(k_1,\dots,k_{2\nu-1}) \\
&= k_{2\nu}\Big(r(k_1,\dots,k_{2\nu-2},k_{2\nu-1}) + s(k_1,\dots,k_{2\nu-2},k_{2\nu-1}) - s(k_1,\dots,k_{2\nu-2})\Big) \\
&\quad + r(k_1,\dots,k_{2\nu-1}) + s(k_1,\dots,k_{2\nu-1}) \\
&= k_{2\nu} s(k_1,\dots,k_{2\nu-2},k_{2\nu-1}) + r(k_1,\dots,k_{2\nu-1}) + s(k_1,\dots,k_{2\nu-1}) \\
&= r(k_1,\dots,k_{2\nu}) + s(k_1,\dots,k_{2\nu}) = r+s.
\end{aligned}$$

- If $N = 2\nu+1$, we follow similar arguments:
  Let D be the matrix of the chain

$$\underbrace{2\cdots2}_{k_1-1}\ (k_2+2)\ \underbrace{2\cdots2}_{k_3-1}\ \cdots\cdots\cdots\ \underbrace{2\cdots2}_{k_{2\nu-3}-1}\ (k_{2\nu-2}+2)$$

then by [5], $k(S) = \delta(k_1, \dots, k_{2\nu+1})$ and

$$\delta(k_1,\dots,k_{2\nu+1}) = \left|\begin{array}{cccccccc}
\boxed{D} & & & & & & & \\
 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & \ddots & \ddots & & & & \\
 & & \ddots & 2 & -1 & & & \\
 & & & \ddots & k_{2\nu}+2 & \ddots & & \\
 & & & & -1 & 2 & \ddots & \\
 & & & & & \ddots & \ddots & -1 \\
 & & & & & & -1 & 2
\end{array}\right|
\begin{array}{l}
1 \\
\sum_{i=1}^{\nu-1} k_{2i-1} \\
\\
\\
\\
\sum_{i=1}^{\nu} k_{2i-1} \\
\\
\\
\sum_{i=1}^{\nu+1} k_{2i-1}
\end{array}$$

$$
= \quad k_{2\nu}(k_{2\nu+1}+1)\left|\begin{array}{cc}\boxed{D} & \\ & -1 \\ -1 & \boxed{\begin{array}{cccc} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{array}}\end{array}\right| \begin{array}{l} 1 \\ \sum_{i=1}^{\nu-1} k_{2i-1} \\ \sum_{i=1}^{\nu-1} k_{2i-1}+1 \\ \\ \sum_{i=1}^{\nu} k_{2i-1}-1 \end{array}
$$

$$
+ \left|\begin{array}{cc}\boxed{D} & \\ & -1 \\ -1 & \boxed{\begin{array}{cccc} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{array}}\end{array}\right| \begin{array}{l} 1 \\ \sum_{i=1}^{\nu-1} k_{2i-1} \\ \sum_{i=1}^{\nu-1} k_{2i-1}+1 \\ \\ \sum_{i=1}^{\nu} k_{2i+1} \end{array}
$$

$$
\begin{aligned}
&= k_{2\nu}(k_{2\nu+1}+1)\delta(k_1,\dots,k_{2\nu-2},k_{2\nu-1}-1)+\delta(k_1,\dots,k_{2\nu-2},k_{2\nu-1}+k_{2\nu+1}) \\
&= k_{2\nu}(k_{2\nu+1}+1)\Big(r_{2\nu-1}+s_{2\nu-1}-s_{2\nu-2}\Big)+s_{2\nu-2}+r_{2\nu-2} \\
&\quad +(k_{2\nu-1}+k_{2\nu+1})s_{2\nu-2}.
\end{aligned}
$$

A straightforward computation show that this last expression is equal to $r_{2\nu+1}+s_{2\nu+1}$.

□

Corollary 4. 25 *The index of the surface $S(G)$ is*

$$
Index(S)=\frac{r+s-1}{gcd\{r+s-1,p+q+l-1\}}.
$$

Corollary 4. 26 *Suppose that $l=d+k(r+s-1)$, then S admits a non trivial global vector field if and only if*

$$
1-\delta(r+s)a_0^{(k+1)r-p+1}=0.
$$

Proof: If $l=d+k(r+s-1)$, it is easy to check that

$$
u=\frac{p+s+rl-1-\delta}{r+s-1}=(k+1)r-p+1.
$$

By propositions 4.23 and 4.24, we have the result. □

Notations 4. 27 *We denote by $\mathcal{G}=\mathcal{G}(p,q,r,s,l)$ the family of contracting birational mappings*

$$
G(z)=\left(z_1^{p+rl}z_2^{q+sl}+\sum_{i=0}^{l-1}a_i\big(z_1^rz_2^s\big)^{i+1}+a_{l+K}\big(z_1^rz_2^s\big)^{l+K+1},z_1^rz_2^s\right),
$$

where $K=\max\left\{0,\left[\frac{l-d}{r+s-1}\right]\right\}$, $a_0\in\mathbb{C}^\star$, $a_i\in\mathbb{C}$, $i=1,\dots,l-1,l+K$, and by $\Phi=\Phi(p,q,r,s,l)$ the group of germs of biholomorphisms $\varphi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ for which there exists $G,G'\in\mathcal{G}$ such that $G'=\varphi^{-1}G\varphi\in\mathcal{G}$. Let $L:=L(p,q,r,s,l)$ be the group of diagonal linear mappings $\varphi_{A,B}(z_1,z_2)=(Az_1,Bz_2)$ where A,B satisfy the condition

$$
B=A^rB^s,\quad A=A^{p+rl}B^{q+sl}
$$

Lemma 4. 28 *1) The group L is a subgroup of $\mathbb{U}_{p+s+rl-\delta-1}\times\mathbb{U}_{p+s+rl-\delta-1}$, where for any $m\in\mathbb{N}^\star$, $\mathbb{U}_m$ is the group of m-roots of unity.*
2) The group L operates on $\mathcal{G}$; more precisely if $\varphi_{A,B}\in L$ and

$$G(z)=\left(z_1^{p+rl}z_2^{q+sl}+\sum_{i=0}^{l-1}a_i\big(z_1^rz_2^s\big)^{i+1}+a_{l+K}\big(z_1^rz_2^s\big)^{l+K+1},z_1^rz_2^s\right),$$

then

$$G'(z)=\varphi_{A,B}^{-1}G\varphi_{A,B}(z)=\left(z_1^{p+rl}z_2^{q+sl}+\sum_{i=0}^{l-1}a'_i\big(z_1^rz_2^s\big)^{i+1}+a'_{l+K}\big(z_1^rz_2^s\big)^{l+K+1},z_1^rz_2^s\right),$$

where

$$Aa'_i=B^{i+1}a_i,\quad \text{for}\quad i=0,\dots,l-1,l+K.$$

In particular L is an abelian subgroup of Φ.

The proof is easy and left to the reader. □

4.1.2 Moduli spaces of birational mappings

We want to determine the equivalence classes of the birational mappings G, or, that is equivalent, the fibers of the canonical morphism to the OT moduli space. Let

$$G(z)=\Pi\sigma(z)=\left(z_1^{p+rl}z_2^{q+sl}+\sum_{i=0}^{l-1}a_i\big(z_1^rz_2^s\big)^{i+1}+a_{l+K}\big(z_1^rz_2^s\big)^{l+K+1},z_1^rz_2^s\right),$$

$$G'(z)=\Pi'\sigma'(z)=\left(z_1^{p+rl}z_2^{q+sl}+\sum_{i=0}^{l-1}a'_i\big(z_1^rz_2^s\big)^{i+1}+a'_{l+K}\big(z_1^rz_2^s\big)^{l+K+1},z_1^rz_2^s\right)$$

be two such birational germs and suppose that there exists a germ of biholomorphism φ such that $G'\circ\varphi=\varphi\circ G$. Since the degeneration set $\{z_1z_2=0\}$ is invariant and φ cannot swap the rational curves, φ has the form

$$\varphi(z_1,z_2)=\big(Az_1(1+\theta(z)),\ Bz_2(1+\mu(z))\big).$$

We have

$$\varphi(G(z))=\left(A\Big(z_1^{p+rl}z_2^{q+sl}+\sum_{i\in\{0,\dots,l-1,l+K\}}a_i\big(z_1^rz_2^s\big)^{i+1}\Big)\big(1+\theta(G(z))\big),Bz_1^rz_2^s\big(1+\mu(G(z))\big)\right)$$

$$\begin{aligned}G'(\varphi(z)) \quad = \quad & \Big(\Big[Az_1(1+\theta(z))\Big]^{p+rl}\Big[Bz_2(1+\mu(z))\Big]^{q+sl}\\ &+\sum_{i\in\{0,\dots,l-1,l+K\}}a'_i\Big(Az_1(1+\theta(z))\Big)^{r(i+1)}\Big(Bz_2(1+\mu(z))\Big)^{s(i+1)},\\ &\qquad\qquad A^rB^sz_1^rz_2^s(1+\theta(z))^r(1+\mu(z))^s\Big)\end{aligned}$$

Second members give the equality

$$(II)\qquad B\big(1+\mu(G(z))\big)=A^rB^s(1+\theta(z))^r(1+\mu(z))^s$$

Therefore

$$B=A^rB^s,\quad\text{and}\quad 1+\mu(z)=\left(\prod_{j=0}^{\infty}\Big(1+\theta(G^j(z))\Big)^{r/s^{j+1}}\right)^{-1}. \tag{3}$$

First members of the conjugation give

$$(I)\qquad \begin{cases} A\Big(z_1^{p+rl}z_2^{q+sl}+\displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i\big(z_1^r z_2^s\big)^{i+1}\Big)\big(1+\theta(G(z))\big) \\ \qquad = \Big(Az_1(1+\theta(z))\Big)^{p+rl}\Big(Bz_2(1+\mu(z))\Big)^{q+sl} \\ \qquad\quad + \displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i'\Big(Az_1(1+\theta(z))\Big)^{r(i+1)}\Big(Bz_2(1+\mu(z))\Big)^{s(i+1)} \end{cases}$$

Setting $\delta = ps - qr = \pm 1$, we obtain with (3),

$$(I)\qquad \begin{cases} A\Big(z_1^{p+rl}z_2^{q+sl}+\displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i\big(z_1^r z_2^s\big)^{i+1}\Big)\big(1+\theta(G(z))\big) \\ \qquad = A^{p+rl}B^{q+sl}z_1^{p+rl}z_2^{q+sl}(1+\theta(z))^{\delta/s}\left(\displaystyle\prod_{j=1}^{\infty}\Big(1+\theta(G^j(z))\Big)^{r/s^{j+1}}\right)^{-(q+sl)} \\ \qquad\quad + \displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i' B^{i+1}(z_1^r z_2^s)^{(i+1)}\left(\displaystyle\prod_{j=1}^{\infty}\Big(1+\theta(G^j(z))\Big)^{r/s^{j+1}}\right)^{-s(i+1)} \end{cases}$$

Lemma and Definition 4. 29 *With the previous notations, given p, q, r, s, l, the positive integral solutions (i, j) of the system*

$$(E)\qquad \begin{cases} p+rl+i &= r\gamma \\ q+sl+j &= s\gamma \end{cases}$$

for which there exists $\gamma \geq 1$ are all of the form

$$\begin{cases} i &= kr-p \\ j &= ks-q \end{cases},\qquad k\geq 1.$$

We have then $\gamma = k + l$. In particular the least solution is $(r - p, s - q)$. When (E) has a solution we shall say that there is a **resonance***.*

Proof: We have $\gamma = l + \frac{p+i}{r} = l + \frac{q+j}{s}$. Since $\frac{p+i}{r}$ and $\frac{q+j}{s}$ are integers, there exists $k, k' \in \mathbb{N}$ such that $p + i = kr$ and $q + j = k's$. Moreover $k = \frac{p+i}{r} = \frac{q+j}{s} = k'$ which gives the result. The other assertions are evident. $\square$

Comparing monomial terms $z_1^{p+rl}z_2^{q+sl}$ in (I) we obtain thanks to lemma 4.29

$$A = A^{p+rl}B^{q+sl} \tag{4}$$

By lemma 4.28, A, B are roots of unity.
Let $Aut(\mathbb{C}^2, 0)$ be the group of germs of biholomorphisms of $(\mathbb{C}^2, 0)$ and $Aut(\mathbb{C}^2, H, 0)$ be the subgroup of $Aut(\mathbb{C}^2, 0)$ whose germs leave each of the components of the hypersurface $H = \{z_1z_2 = 0\}$ invariant, i.e. $\varphi \in Aut(\mathbb{C}^2, H, 0)$ has the form

$$\varphi(z) = \big(Az_1(1+\theta(z)), Bz_2(1+\mu(z))\big).$$

Notice that $\Phi \subset Aut(\mathbb{C}^2, H, 0)$.
Let $Aut_{Id}(\mathbb{C}^2, 0)$ (resp. $Aut_{Id}(\mathbb{C}^2, H, 0)$) be the subgroup of $Aut(\mathbb{C}^2, 0)$ (resp. $Aut(\mathbb{C}^2, H, 0)$) of germs of biholomorphisms φ tangent to the identity, i.e. $\varphi \in Aut_{Id}(\mathbb{C}^2, H, 0)$ if

$$\varphi(z) = \big(z_1(1+\theta(z)), z_2(1+\mu(z))\big).$$

Lemma 4. 30 *Let $\alpha : Aut_{Id}(\mathbb{C}^2,H,0) \to Aut(\mathbb{C}^2,H,0)$ the canonical injection and $\beta : Aut(\mathbb{C}^2,H,0) \to L$ defined by $\beta(\varphi) = \varphi_{AB}$ the linear part of φ. Then, $Aut_{Id}(\mathbb{C}^2,H,0)$ is a normal subgroup of $Aut(\mathbb{C}^2,H,0)$ and we have the exact sequence*

$$\{Id\} \to Aut_{Id}(\mathbb{C}^2,H,0) \xrightarrow{\alpha} Aut(\mathbb{C}^2,H,0) \xrightarrow{\beta} L \to \{Id\}.$$

Replacing φ by $\varphi\varphi_{A,B}^{-1}$ we obtain an automorphism tangent to the identity, therefore we have to determine equivalence classes of the equivalence relation on $\mathcal{G}$

$$G \sim G' \iff \exists\, \varphi \in Aut_{Id}(\mathbb{C}^2,H,0), \quad G'\varphi = \varphi G.$$

The equation (I) becomes

$$(I) \qquad \begin{cases} \left(z_1^{p+rl}z_2^{q+sl} + \displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i\left(z_1^rz_2^s\right)^{i+1}\right)\left(1+\theta(G(z))\right) \\[2ex] \quad = z_1^{p+rl}z_2^{q+sl}(1+\theta(z))^{\delta/s}\left(\displaystyle\prod_{j=1}^{\infty}\left(1+\theta(G^j(z))\right)^{r/s^{j+1}}\right)^{-(q+sl)} \\[2ex] \quad + \displaystyle\sum_{i\in\{0,\dots,l-1,l+K\}} a_i'(z_1^rz_2^s)^{i+1}\left(\displaystyle\prod_{j=1}^{\infty}\left(1+\theta(G^j(z))\right)^{r/s^{j+1}}\right)^{-s(i+1)} \end{cases}$$

Notice that we obtain immediately $a_0 = a_0'$.
The question is to determine the quotient $\mathcal{G}/\sim$. We shall see at the end of this section that the equivalence relation is generically trivial.

In the following lemma the maximum is due to the fact that we may have $l-d<0$.

Lemma 4. 31 *Let $\mu = \max\left\{d, l+\left[\dfrac{l-d}{r+s-1}\right]\right\}$ and $\theta(z) = \sum_{i+j\geq 1} t_{ij}z_1^iz_2^j$. If $t_{ij}=0$ for $i+j\leq\mu$, then*

$$\theta = 0$$

and φ is linear.

Proof: By hypothesis we have

$$\theta(G(z)) = \left(\sum_{i+j=\mu+1} a_0^it_{ij}\right)(z_1^rz_2^s)^{\mu+1} \mod \mathfrak{M}^{(r+s)(\mu+1)+1},$$

hence

$$\prod_{j=1}^{\infty}\left(1+\theta(G^j(z))\right)^{r/s^{j+1}} = 1+\frac{r}{s^2}\left(\sum_{i+j=\mu+1} a_0^it_{ij}\right)(z_1^rz_2^s)^{\mu+1} \mod \mathfrak{M}^{(r+s)(\mu+1)+1}.$$

We show by induction on $k=i+j\geq\mu+1$ that $t_{ij}=0$.
We consider the terms of degree $p+q+(r+s)l+\mu+1$

$$z_1^{p+rl}z_2^{q+sl}\,\frac{\delta}{s}\sum_{i+j=\mu+1} t_{ij}z_1^iz_2^j$$

and we are looking for other terms of the same degree or bidegrees in (I).

The inequalities

$$\begin{cases} p+q+(r+s)l+\mu+1 < p+q+(r+s)l+(r+s)(\mu+1), \\ p+q+(r+s)l+\mu+1 < r+s+(r+s)(\mu+1) \end{cases}$$

show that there is no other term of the same degree when $a_{K+l} = 0$. If $a_{K+l} \neq 0$, $l = d + K(r+s-1)$ and it is easy to check that $(r+s)(l+K) \neq p+q+(r+s)l+\mu+1$, hence $t_{ij} = 0$ if $i+j = \mu+1$.
Suppose that for $k \geq \mu+2$,

$$\theta(z) = \sum_{i+j\geq k} t_{ij} z_1^i z_2^j,$$

then the similar inequalities show the result. □

Lemma 4. 32 *Let* $\mu = \max\left\{d, l + \left[\frac{l-d}{r+s-1}\right]\right\}$.
Then, the coefficients t_{ij}*, for* $i+j \leq \mu$*, with* a_i *and* a'_i*,* $i = 0, \dots, l-1, l+K$*, determine uniquely* θ *hence also* φ*.*

Proof: We show by induction on $k \geq 0$ that the coefficients t_{ij} for $i+j \leq \mu$ determine uniquely the coefficients t_{ij} for $i+j \geq \mu+k$. It is sufficient to show that if the coefficients t_{ij}, for $i+j \leq \mu+k$ are determined by coefficients t_{ij} for $i+j \leq \mu$ then the coefficients t_{ij} for $i+j = \mu+k+1$ are determined by coefficients t_{ij} for $i+j \leq \mu+k$. On that purpose we consider homogeneous part of degree $p+q+(r+s)l+\mu+k+1$ in (I) which contains the part

$$z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s}\left(\sum_{i+j=\mu+k+1} t_{ij} z_1^i z_2^j\right)$$

In order to prove that all other terms with such degree involve only t_{ij} with $i+j \leq \mu+k$, it is sufficient to prove that if $i+j \geq \mu+k+1$ then

$$r+s+(i+j)(r+s) > p+q+(r+s)l+\mu+k+1,$$

and it is sufficient to prove that

$$r+s+(\mu+k+1)(r+s) > p+q+(r+s)l+\mu+k+1.$$

- If $l \leq d$, $\mu = d$, and we have to check that

  $$r+s+(d+k+1)(r+s) > p+q+(r+s)d+d+k+1$$

  which is clear;
- If $d+K(r+s-1) \leq l < d+(K+1)(r+s-1)$, then $\mu = l+K$. We have to check

  $$r+s+(l+K+k+1)(r+s) > p+q+(r+s)l+l+K+k+1$$

  However this inequality is equivalent to

  $$d+(K+k+1)(r+s-1) > l$$

  which is satisfied by assumption.

□

Proposition 4. 33 *Let $\mathcal{G} = \mathcal{G}(p,q,r,s,l)$ the family of contracting birational mappings*

$$G(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i \big(z_1^r z_2^s\big)^{i+1} + a_{l+K} \big(z_1^r z_2^s\big)^{l+K+1}, z_1^r z_2^s \right),$$

where $K = \max\left\{0, \left[\frac{l-d}{r+s-1}\right]\right\}$, $a_0 \in \mathbb{C}^\star$, $a_i \in \mathbb{C}$, $i = 1, \ldots, l-1, l+K$.
1) The group $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi$ is isomorphic to $(\mathbb{C}, +)$ and θ is determined by exactly one coefficient of the homogeneous part of degree $l + K$.
2) Suppose that $\frac{l-d}{r+s-1}$ is a non negative integer, i.e. $l = d + K(r + s - 1)$, then

a) If there are global vector fields, $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi$ acts trivially on $\mathcal{G}$, in particular a_{l+K} is an effective parameter,

b) If there are no global vector fields, $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi$ acts transitively on $\mathbb{C}_{a_{l+K}}$, i.e. the complex structure on $S(G)$ does not depend on a_{l+K}.

3) Suppose that $\frac{l-d}{r+s-1}$ is not a non negative integer, then $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi$ acts transitively on $\mathbb{C}_{a_{l+K}}$, i.e. the complex structure on $S(G)$ does not depend on a_{l+K}.

Proof: Suppose that $\theta \neq 0$ and let $\gamma = \min\{i + j \geq 1 \mid t_{ij} \neq 0\}$. By lemma 31, $\gamma \leq \mu$. The homogeneous parts of lower degree in (I) which involve t_{ij} with $\gamma = i + j$ are

- Case $\gamma \leq l - 1$ or $\gamma = l + K$,

$$(A) \qquad a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma + a_\gamma (z_1^r z_2^s)^{\gamma+1},$$

$$(B) \qquad z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} \sum_{i+j=\gamma} t_{ij} z_1^i z_2^j,$$

$$(C) \qquad -\frac{r}{s} a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma + a'_\gamma (z_1^r z_2^s)^{\gamma+1}$$

- Case $\gamma \geq l$ and $\gamma \neq l + K$, (A) is replaced by

$$(A') \qquad a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma,$$

  and (C) by

$$(C') \qquad -\frac{r}{s} a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma$$

- If there is no resonance, the bidegrees of the terms (A) and (C) (resp. (A') and (C')) are all distinct of those in (B), therefore we obtain readily

$$\sum_{i+j=\gamma} t_{ij} z_1^i z_2^j = 0,$$

  hence a contradiction

- Therefore there is a resonance and there exists a unique coefficient $t_{kr-p,ks-q} \neq 0$ with $k(r+s)-(p+q)=\gamma$. Then
  - Case $\gamma \leq l-1$ or $\gamma = l+K$,

$$a_0 z_1^r z_2^s t_{kr-p,ks-q} a_0^{kr-p}(z_1^r z_2^s)^\gamma + a_\gamma (z_1^r z_2^s)^{\gamma+1}$$
$$= z_1^{p+rl} z_2^{q+sl} \tfrac{\delta}{s} t_{kr-p,ks-q} z_1^{kr-p} z_2^{ks-q} + a'_\gamma (z_1^r z_2^s)^{\gamma+1} - \tfrac{r}{s} a_0 z_1^r z_2^s t_{kr-p,ks-q} a_0^{kr-p}(z_1^r z_2^s)^\gamma$$

  - Case $\gamma \geq l$, $\gamma \neq l+K$

$$a_0 z_1^r z_2^s t_{kr-p,ks-q} a_0^{kr-p}(z_1^r z_2^s)^\gamma$$
$$= z_1^{p+rl} z_2^{q+sl} \tfrac{\delta}{s} t_{kr-p,ks-q} z_1^{kr-p} z_2^{ks-q} - \tfrac{r}{s} a_0 z_1^r z_2^s t_{kr-p,ks-q} a_0^{kr-p}(z_1^r z_2^s)^\gamma$$

The equality of degrees implies that $l = d + (k-1)(r+s-1)$, i.e. the surface has twisted vector fields and $\gamma = l + (k-1) = l + K$ (and the second case never appears). After simplification (recall that $\delta = \pm 1$), we obtain

$$a'_{l+K} = a_{l+K} - t_{kr-p,ks-q} \frac{\delta}{s}\Big(1 - \delta(r+s) a_0^{kr-p+1}\Big)$$

Let $\mathfrak{M} = (z_1, z_2)$. Since $\theta(z) = 0 \mod \mathfrak{M}^{l+K}$, (I) gives

$$a'_i = a_i, \quad \text{for} \quad i = 0, \ldots, l-1,$$

therefore, applying corollary 4.26,

- If there are global vector fields, $1-\delta(r+s)a_0^{kr-p+1} = 0$ and $a'_{l+K} = a_{l+K}$, hence $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi$ acts trivially;
- If there are no vector fields, $1 - \delta(r+s)a_0^{kr-p+1} \neq 0$, and $G_I \cap \Phi$ acts transitively on the line $a_{l+K} \in \mathbb{C}$.

By lemma 4.32, $t = t_{(K+1)r-p,(K+1)s-q} \in \mathbb{C}$ determines the formal series θ. It remains to prove that θ is convergent hence $Aut_{Id}(\mathbb{C}^2, H, 0) \cap \Phi \simeq \mathbb{C}$.

- If there are global vector fields, there exists a 1-parameter group of automorphisms, therefore there are such θ and conversely, any θ defines an automorphism of $S(G)$ which is in the identity component of $Aut(S(G))$;
- If there is no global vector fields, a_{l+K} is a superfluous parameter and all surfaces are isomorphic, therefore there are such isomorphisms.

□

4.1.3 The twisting coefficient

We want to compare birational germs and Favre polynomial germs of the form

$$F(z_1, z_2) = (\lambda z_1 z_2^\sigma + P(z_2) + c z_2^{\frac{\sigma k}{k-1}}, z_2^k), \quad P(z_2) = \sum_{i=p+q}^{\sigma} b_i z_2^i$$

given in [21] (see section 3.3). The parameter λ determines the twisting coefficient κ such that $H^0(S, K_S^{-\mu} \otimes L^\kappa) \neq 0$. In birational germs this role is played by the position of the blown-up point $(a_0, 0)$, $a_0 \neq 0$, on the root of the branch when there is only one branch. The condition $j < k$ (or $p+q < r+s$ in our notations) implies that the first blowing-up is of the form $(u', v') \mapsto (v', u'v')$ hence we have to consider the germ $\Pi_l \cdots \Pi_{n-1} \bar{\sigma} \Pi_0 \cdots \Pi_{l-1}$ at the point $(a_0, 0)$. After a change of coordinates $u = z_1 + a_{l-1}$, $v = z_2$ we obtain

$$G(z_1, z_2) = \left(\Big(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1} \Big)^p z_2^q, \ \Big(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1} \Big)^r z_2^s \right)$$

In case there is no global vector fields, a_{l+K} is a superfluous parameter, hence we shall suppose that $a_{l+K}=0$.

Remark 4. 34 *1) If index$(S)=1$, we have $\lambda^{-1}=k(S)\kappa$, i.e. the invariant used here is the inverse of the invariant $\lambda=\lambda(S)$ in [8].*
2) If index$(S)\neq 1$, $\lambda(a)$ is defined up to a $(k-1)$-root of unity.

Proposition 4. 35 *Let S be a surface with GSS associated to the germ*

$$G(z_1,z_2)=\left(\Bigl(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}+a_{l+K}z_2^{l+K+1}\Bigr)^pz_2^q,\ \Bigl(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}+a_{l+K}z_2^{l+K+1}\Bigr)^rz_2^s\right),\quad a_0\neq 0$$

Let $\mu=index(X)$ be the index of S. Then on the corresponding base $B_{J,M}$ of the family $\Phi_{J,M,\sigma}:\mathcal{S}_{J,M,\sigma}\to B_{J,M}$, the holomorphic function

$$\kappa=\kappa_{J,M,\sigma}:\mathbb{C}^\star\to\mathbb{C}^\star$$

such that $H^0(S_a,K_{S_a}^{-\mu}\otimes L^{\kappa(a)})\neq 0$ is a monomial holomorphic function of a_0, where $O_0=(a_0,0)$. More precisely, if $\delta=ps-qr$ and $\sigma=p+q+l-1$,

$$\kappa_{J,M,\sigma}(a_0)=\delta^\mu a_0^{\mu\left(\frac{r\sigma}{r+s-1}-p+1\right)},\quad \mu\left(\frac{r\sigma}{r+s-1}-p+1\right)\in\mathbb{N}^\star.$$

In particular κ is surjective.

Proof: Let $\kappa:=\kappa_{J,M,\sigma}$ be the holomorphic function given by [6] prop.4.24.
Setting

$$\left\{\begin{array}{l}\Bigl(\ \Bigr):=z_1z_2^l+\displaystyle\sum_{i\in\{0,\ldots,l-1,l+K\}}a_iz_2^{i+1}\quad\text{and}\\[2em] \Bigl[\ \Bigr]:=\dfrac{\partial}{\partial z_2}\Bigl(\ \Bigr)=lz_1z_2^{l-1}+\displaystyle\sum_{i\in\{0,\ldots,l-1,l+K\}}a_i(i+1)z_2^i\end{array}\right.$$

$$DG(z)=\begin{pmatrix}p\Bigl(\ \Bigr)^{p-1}z_2^{l+q} & p\Bigl(\ \Bigr)^{p-1}\Bigl[\ \Bigr]z_2^q+\Bigl(\ \Bigr)^pqz_2^{q-1}\\[1em] r\Bigl(\ \Bigr)^{r-1}z_2^{l+s} & r\Bigl(\ \Bigr)^{r-1}\Bigl[\ \Bigr]z_2^s+\Bigl(\ \Bigr)^rsz_2^{s-1}\end{pmatrix}$$

and

$$\det DG(z)=(ps-qr)\Bigl(\ \Bigr)^{p+r-1}z_2^{l+q+s-1}.$$

Let $\theta\in H^0(S,K_S^{-\mu}\otimes L^\kappa)$, then there exists an invariant germ in a neighbourhood of the origin of the ball still denoted by θ which vanishes only on the curve $\{z_2=0\}$,

$$\theta(z)=z_2^\alpha A(z)\left(\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_1}\right)^{\otimes\mu}$$

such that $A(0)\neq 0$. This germ satisfies the condition

$$\theta(G(z))=\kappa\bigl(\det DG(z)\bigr)^\mu\theta(z),$$

which is equivalent to

$$\Bigl(\ \Bigr)^{\alpha r}z_2^{\alpha s}A(G(z))=\kappa(ps-qr)^\mu\Bigl(\ \Bigr)^{\mu(p+r-1)}z_2^{\mu(l+q+s-1)+\alpha}A(z)$$

where $\delta := ps - qr = \pm 1$. Considering the homogeneous part of lower degree of each member, we obtain

$$\alpha(r+s-1) = \mu(p+q+l-1+r+s-1) = \mu(\sigma+r+s-1) \tag{1}$$

$$\kappa = (ps-qr)^{\mu} a_0^{\alpha r - \mu(p+r-1)} \tag{2.}$$

By [6] proposition 4.24, κ vanishes on smaller strata, i.e. when $a_0 = 0$, therefore $\alpha r - \mu(p+r-1) > 0$. We derive the value of κ from (1) and (2). □

4.1.4 Representation of surfaces with one branch and without twisted vector fields by birational germ

We suppose that $\boldsymbol{l - d \not\equiv 0 \mod r+s-1}$, $\boldsymbol{a_{l+K} = 0}$.
Given $F(z_1, z_2) = \left(\lambda z_1 z_2^{\sigma} + \sum_{i=p+q}^{\sigma} b_i z_2^i,\ z_2^{r+s}\right)$, there exist germs G which have the same twisting coefficient λ as F by the surjectivity of κ (Prop. 4.35). The aim of the sequel of this section is to prove

Theorem 4. 36 *Given λ, we choose suitably $a_0 \in \mathbb{C}^{\star}$ and ϵ, $\epsilon^{r+s-1} = 1$, in such a way that any $G \in \mathcal{G}(p,q,r,s,l)$ with parameter a_0 is conjugated to a germ $F \in \mathcal{F}(\sigma, k, j)$ with parameter λ. Let $\sigma = p+q+l-1$. Then*
A) If $r+s-1$ does not divide $l-d$ or $\lambda \neq 1$ there is a bijective polynomial mapping

$$\begin{array}{cccc} f_{a_0,\epsilon}: & \mathbb{C}^{l-1} & \longrightarrow & \mathbb{C}^{l-1} \\ & a = (a_1, \ldots, a_{l-1}) & \longmapsto & \Big(b_{p+q+1}(a), \ldots, b_{p+q+l-1}(a)\Big) \end{array}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1}\right)^p z_2^q,\ \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1}\right)^r z_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^{\sigma} + \sum_{i=p+q}^{\sigma} b_i z_2^i,\ z_2^{r+s}\right),$$

where λ as κ depend only on a_0 by 4.35 and $b_{p+q} = 1$.
B) If $l-d = K(r+s-1)$ and $\lambda = 1$, there is a bijective polynomial mapping

$$\begin{array}{cccc} f_{a_0,\epsilon}: & \mathbb{C}^{l-1} \times \mathbb{C} & \longrightarrow & \mathbb{C}^{l-1} \times \mathbb{C} \\ & a = (a_1, \ldots, a_{l-1}, a_{l+K}) & \longmapsto & \Big(b_{p+q+1}(a), \ldots, b_{p+q+l-1}(a), c(a)\Big) \end{array}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^p z_2^q,\ \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^r z_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^{\sigma} + \sum_{k=p+q}^{\sigma} b_k z_2^k + c z_2^{\frac{\sigma k(S)}{k(S)-1}}, z_2^{r+s}\right),$$

where $b_{p+q} = 1$.

Proof: Let $\varphi(z) = \big(\varphi_1(z), Cz_2(1+\mu(z))\big)$ be a germ of biholomorphic map which preserves the degeneration set $\{z_2 = 0\}$.

$$\varphi(G(z)) = \left(\varphi_1\Big(G(z)\Big),\ C\left\{z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1}\right\}^r z_2^s\Big(1+\mu(G(z)\Big)\right),$$

$$F(\varphi(z)) = \left(\lambda\varphi_1(z)C^\sigma z_2^\sigma(1+\mu(z))^\sigma + \sum_{k=p+q}^{\sigma} b_k C^k z_2^k(1+\mu(z))^k,\ C^{r+s}z_2^{r+s}(1+\mu(z))^{r+s}\right).$$

Comparing the right members we have

$$(II) \qquad \left\{z_1 z_2^{l-1} + \sum_{i=0}^{l-1} a_i z_2^i\right\}^r \Big(1+\mu(G(z)\Big) = C^{r+s-1}(1+\mu(z))^{r+s}.$$

Constant parts give the condition

$$a_0^r = C^{r+s-1} \tag{5}$$

therefore C is determined up to a root of unity ϵ such that $\epsilon^{r+s-1} = 1$. In other terms if we choose a local determination of the $(r+s-1)$-root $a_0^{1/(r+s-1)}$,

$$C = \epsilon a_0^{r/(r+s-1)}, \quad \epsilon^{r+s-1} = 1. \tag{6}$$

Moreover the equation

$$(II) \qquad \left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^r \Big(1+\mu(G(z)\Big) = (1+\mu(z))^{r+s}.$$

has the solution

$$1+\mu(z) = \prod_{j=0}^{\infty}\left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i (G_2^j(z))^i + G_1^j(z)(G_2^j(z))^{l-1}\right)\right\}^{\frac{r}{(r+s)^{j+1}}}$$

Left members give the equality

$$(I) \quad \begin{cases} \varphi_1\left(a_0^p\left\{1+\frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^p z_2^{p+q},\ a_0^r\left\{1+\frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^r z_2^{r+s}\right) \\ \qquad = \lambda\varphi_1(z)\Big(Cz_2(1+\mu(z)\Big)^{p+q+l-1} + \displaystyle\sum_{k=p+q}^{p+q+l-1} b_k C^k z_2^k(1+\mu(z))^k \end{cases}$$

We want to express the coefficients b_k with the a_i's, however the coefficients A_{ij} of the series

$$\varphi_1(z_1, z_2) = \sum_{i,j} A_{ij} z_1^i z_2^j$$

depend also on a_i's. For example, considering homogeneous parts of bidegree $(0, p+q)$, we have,

$$(R_0) \qquad A_{10}a_0^p = b_{p+q}C^{p+q} = C^{p+q}$$

hence with (5),

$$A_{10} = \epsilon a_0^{\frac{p-\delta}{r+s-1}}. \tag{7}$$

If $p > 0$, $r+s > p+q+1$ and $l \geq 2$, homogeneous part of bidegree $(0, p+q+1)$ gives

$$A_{10}a_0^{p-1}pa_1 = b_{p+q+1}C^{p+q+1} + C^{p+q}\frac{(p+q)r}{r+s}\frac{a_1}{a_0}$$

therefore by (R_0),

$$(R_1) \qquad b_{p+q+1} = \frac{\delta a_1}{Ca_0(r+s)}.$$

Comparing terms of bidegree $(1, p+q+l-1)$ we obtain

$$A_{10} p a_0^{p-1} = \lambda A_{10} C^{p+q+l-1} + \frac{r(p+q)}{r+s} \frac{C^{p+q}}{a_0}$$

therefore with (7) and (6), and since $k = k(S) = r+s$,

$$\lambda = \frac{\delta}{\epsilon^\sigma k} a_0^{p-1-\frac{r\sigma}{k-1}}. \tag{8}$$

where $\delta = ps - qr$ (with formula $\lambda^{-1} = \kappa k$, notice that we recover the result of Prop 4.35) .

In order to express the coefficients b_{p+q+j}, $j \geq 1$, as polynomials of variables $a_1, \ldots, a_{l-1}$, it is also necessary to express the coefficients A_{ij} involved in the relations as polynomials of the same variables $a_1, \ldots, a_{l-1}$. Therefore we have to determine the set of points $(i,j) \in \mathbb{N} \times \mathbb{N}$ which occur as indices of the A_{ij}'s in the relations.
Let E_0 be the subset of indices (i,j) which occur in homogeneous part of bidegree $(0,k)$ for $p+q \leq k \leq p+q+l-1$ in equation (I). We have

$$E_0 = \{(i,j) \mid p+q \leq i(p+q) + j(r+s) \leq p+q+l-1\}$$

Then we define a translation

$$T(i,j) = (i, j+p+q+l-1)$$

and we want to determine which coefficients $A_{\alpha\beta}$ are involved on the homogeneous part of bidegree $T(i,j)$. On that purpose we define a sequence $(E_m)_{m \geq -1}$ of increasing subsets of $\mathbb{N} \times \mathbb{N}$, starting with $E_{-1} = \emptyset$,

$$E_m = \left\{(i,j) \mid i(p+q) + j(r+s) \leq (p+q+l-1)\left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^m}\right)\right\}, \quad m \geq 0,$$

and

$$E_\infty := \left\{(i,j) \mid i(p+q) + j(r+s) < (p+q+l-1)\frac{r+s}{r+s-1}\right\}.$$

For any polynomial expression $Q(z_1, z_2)$ we denote by $\langle Q(z_1,z_2)\rangle_{a,b}$ the homogeneous part of bidegree (a,b).

Lemma 4. 37 *Suppose $l - d \not\equiv 0$ mod $r+s-1$. Let $(i,j) \in E_m$, $m \geq 0$.*
1) If $i \geq 2$ then for any (α, β), the homogeneous parts of bidegree $(i, j+p+q+l-1)$ satisfy

$$\left\langle A_{\alpha\beta}\ a_0^{p\alpha + r\beta} \left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)}\right\rangle_{i,j+p+q+l-1} = 0.$$

2) If $i = 1$, and homogeneous part of bidegree $(i, j+p+q+l-1)$ satisfies

$$\left\langle A_{\alpha\beta}\ a_0^{p\alpha + r\beta} \left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)}\right\rangle_{i,j+p+q+l-1} \neq 0$$

then $(\alpha, \beta) \in E_1$.
Moreover

- *If $m = 0$, then $(\alpha, \beta) \in E_0$,*
- *If $\alpha = 1$, then $\beta \leq j/(r+s)$, in particular if $j \neq 0$, then $\beta \neq j$,*
- *If $\alpha = 0$, then $\beta < j$ or $\Big\{(i,j) = (1,1) \text{ and } (\alpha,\beta) = (0,1)\Big\}$.*

3) If $i = 0$ and

$$\left\langle A_{0\beta}\ a_0^{r\beta} \left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^{r\beta} z_2^{\beta(r+s)} \right\rangle_{0,j+p+q+l-1} \neq 0,$$

the following two conditions cannot be fulfilled at the same time

- *$i = \alpha = 0$, and $j = \beta$,*
- *$\beta(r+s) = j + p + q + l - 1$,*

i.e. if the coefficient $A_{0,j}$ appears two times when considering homogeneous part of bidegree $(0, j + p + q + l - 1)$, one occurrence is multiplied by a non constant polynomial in $a_1, \ldots, a_j$.
4) If $(i,j) \in E_m$ and $i \geq 2$, then $A_{ij} = 0$.
5) If $(0,j) \in E_m \setminus E_{m-1}$, $m \geq 0$ and (α,β) satisfies

$$\alpha(p+q) + \beta(r+s) = j + p + q + l - 1,$$

then

- *$(\alpha,\beta) \in E_{m+1} \setminus E_m$*
- *$\alpha = 0$ or $\alpha = 1$ and (α,β) is unique.*

In other words, in homogeneous part of bidegree $(0, j + p + q + l - 1)$, there are, modulo $\mathfrak{M} = (a_1, \ldots, a_{l-1})$, at most two coefficients which occur: $A_{0,j}$ and perhaps another $A_{\alpha\beta}$ with $\alpha = 0$ or $\alpha = 1$.

Proof: If

$$\left\langle A_{\alpha\beta}\ a_0^{p\alpha+r\beta} \left\{1 + \frac{1}{a_0}\left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1}\right)\right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)} \right\rangle_{i,j+p+q+l-1} \neq 0$$

then, $p\alpha + r\beta \geq i$, and the least degree in z_2 is

$$(l-1)i + \alpha(p+q) + \beta(r+s).$$

Since $\langle . \rangle_{i,j+p+q+l-1} \neq 0$,

$$(*) \qquad (l-1)i + \alpha(p+q) + \beta(r+s) \leq j + p + q + l - 1$$

however by assumption $(i,j) \in E_m \subset E_\infty$,

$$j < \frac{p+q+l-1}{r+s-1} - i\ \frac{p+q}{r+s}$$

therefore

$$(**) \qquad (l-1)i + i\ \frac{p+q}{r+s} + \alpha(p+q) + \beta(r+s) < (p+q+l-1)\left(1 + \frac{1}{r+s-1}\right)$$

1) If $i \geq 2$, we have, by $(**)$,

$$2(l-1) + 2\ \frac{p+q}{r+s} + \alpha(p+q) + \beta(r+s) < (p+q+l-1)\left(1 + \frac{1}{r+s-1}\right)$$

Since $p+q<r+s$,

$$(l-1)\left(1-\frac{1}{r+s-1}\right)+(p+q)\left(\frac{2}{r+s}-\frac{1}{r+s-1}+\alpha+\beta-1\right)<0$$

which is impossible.
2) Suppose that $(i,j)\in E_m$, and $i=1$ then

$$j<\frac{p+q+l-1}{r+s}\left(1+\frac{1}{r+s-1}\right)-\frac{p+q}{r+s}$$

and by $(*)$

$$(l-1)+\frac{p+q}{r+s}+\alpha(p+q)+\beta(r+s)<(p+q+l-1)\left(1+\frac{1}{r+s}+\frac{1}{(r+s)(r+s-1)}\right)$$

which is equivalent to

$$(l-1)\left(1-\frac{1}{(r+s)(r+s-1)}\right)+(p+q)\left(\frac{1}{r+s}-\frac{1}{(r+s)(r+s-1)}\right)+\alpha(p+q)+\beta(r+s)$$
$$<(p+q+l-1)\left(1+\frac{1}{r+s}\right)$$

hence

$$\alpha(p+q)+\beta(r+s)\leq(p+q+l-1)\left(1+\frac{1}{r+s}\right)$$

and $(\alpha,\beta)\in E_1$.
If $m=0$, the result derives from the definition of E_0 and $(*)$.
If in $(*)$, $\alpha=1$, $\beta(r+s)\leq j$.
If in $(*)$, $\alpha=0$, $\beta(r+s)\leq j+(p+q)$. If moreover $\beta\geq j$, then

- $j=0$ and $\beta(r+s)\leq p+q$ which is impossible because $\alpha=0$,
- $j=1$, and $\beta=1$.

3) Let $(0,j)\in E_m$, with $m\geq 0$ is minimal. We have

$$j(r+s)\leq(p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^m}\right).$$

If $j(r+s)=j+p+q+l-1$, then

$$j\leq(p+q+l-1)\left(\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^m}\right)$$

hence

$$j(r+s)\leq(p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^{m-1}}\right).$$

and $(0,j)\in E_{m-1}$ which is contradictory.
4) Let $(i,j)\in E_m$ with $i\geq 2$ and consider part of bidegree $(i,j+p+q+l-1)$. By 1), left member of (I) gives no contribution, we show now that

$$\left\langle b_kC^kz_2^k(1+\mu(z))^k\right\rangle_{i,j+p+q+l-1}=0.$$

In fact, the monomials which contain z_1^i contain z_2 at the power at least $k+i(l-1)$ with $p+q\leq k\leq p+q+l-1$ and it is sufficient to show that

$$j+p+q+l-1<k+i(l-1).$$

Moreover, $k \geq p+q$ and $i \geq 2$, hence it is sufficient to prove that $j+p+q+l-1 < p+q+2(l-1)$, i.e.

$$(\maltese) \qquad\qquad j < l-1.$$

By assumption, $(i,j) \in E_m$, therefore

$$j \leq \frac{1}{r+s}(p+q+l-1)\left(1+\cdots+\frac{1}{(r+s)^m}\right) - i\frac{p+q}{r+s}$$

and condition $(\maltese)$ is satisfied if

$$\frac{1}{r+s}(p+q+l-1)\left(1+\cdots+\frac{1}{(r+s)^m}\right) < (l-1) + 2\frac{p+q}{r+s}$$

which is clearly satisfied since $r+s \geq 2$. Finally we obtain

$$0 = \lambda A_{ij} C^{p+q+l-1} z_1^i z_2^{j+p+q+l-1}$$

and $A_{ij} = 0$.
5) If $(0,j) \in E_m \setminus E_{m-1}$,

$$(p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^{m-1}}\right) < j(r+s)$$
$$\leq (p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^m}\right)$$

By hypothesis, (α,β) satisfies

$$\alpha(p+q)+\beta(r+s) = j+p+q+l-1$$

therefore the following holds

$$(p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^m}\right) < \alpha(p+q)+\beta(r+s)$$
$$\leq (p+q+l-1)\left(1+\frac{1}{r+s}+\cdots+\frac{1}{(r+s)^{m+1}}\right)$$

i.e. $(\alpha,\beta) \in E_{m+1} \setminus E_m$. By 4), $\alpha = 0$ or $\alpha = 1$.
If $k = (p+q)+\beta(r+s) = \beta'(r+s)$ then $p+q$ is a multiple of $r+s$ which is impossible, therefore we have the unicity of (α,β). □

Lemma 4. 38 *The linear system with coefficients in* $\mathbb{C}[a_1,\ldots,a_{l-1}]$ *and unknowns*

$$b_{p+q+1},\ldots,b_{p+q+l-1} \quad \text{and} \quad A_{ij}, \quad (i,j) \in E_\infty$$

is a Cramer system of order $l-1+\mathrm{Card}\,(E_\infty)$. *More precisely, modulo* $\mathfrak{M}$, *its determinant is*

$$\Delta = C^{p+q+1}\cdots C^{p+q+l-1}(\lambda C^{p+q+l-1})^{\mathrm{Card}\,E_\infty} \neq 0 \mod \mathfrak{M} := (a_1,\ldots,a_{l-1})$$

and $b_k = \frac{B_k}{\Delta}$, $k = p+q+1,\ldots,p+q+l-1$, $A_{ij} = \frac{B_{ij}}{\Delta}$, $(i,j) \in E_\infty$, *with* $B_k, B_{ij} \in \mathbb{C}[a_1,\ldots,a_{l-1}]$. *In particular,* $f_{a_0,\epsilon}$ *is rational.*

Proof: We order the unknowns in the following way: First unknowns $b_{p+q+1},\ldots,b_{p+q+l-1}$, after coefficients $A_{0j} \neq 0$, with $(0,j) \in E_0$ then $(0,j) \in E_1 \setminus E_0$, $\ldots (0,j) \in E_{m+1} \setminus E_m$, exhausting E_∞. Finally coefficients A_{1j}, with j in the decreasing order. We have the same number of equations and

of unknowns, therefore we have a linear system of order $l-1+\mathrm{Card}\,(E_\infty)$. Let $\mathfrak{M}=(a_1,\dots,a_{l-1})$. In order to prove that we have a Cramer system it is sufficient to prove that modulo $\mathfrak{M}$ the determinant Δ is nonzero. Therefore we consider the equation (I) modulo $\mathfrak{M}$, i.e.
$(I_{\mathfrak{M}})$

$$\left\{\begin{aligned} &\varphi_1\left(a_0^p\left\{1+\frac{z_1z_2^{l-1}}{a_0}\right\}^p z_2^{p+q},\ a_0^r\left\{1+\frac{z_1z_2^{l-1}}{a_0}\right\}^r z_2^{r+s}\right)\\ &\qquad\qquad = \lambda\varphi_1(z)\Big(Cz_2(1+\mu(z)\Big)^{p+q+l-1}+\sum_{k=p+q}^{p+q+l-1} b_kC^kz_2^k(1+\mu(z))^k \qquad \mathrm{mod}\ \mathfrak{M}\end{aligned}\right.$$

where, in the infinite product $1+\mu(z)$,

$$\begin{aligned} G(z_1,z_2) &= \Big((z_1z_2^l+a_0z_2)^pz_2^q,\ (z_1z_2^l+a_0z_2)^rz_2^s\Big)\\ &= \left(a_0^p\left(1+\frac{z_1z_2^{l-1}}{a_0}\right)^p z_2^{p+q},\ a_0^r\left(1+\frac{z_1z_2^{l-1}}{a_0}\right)^r z_2^{r+s}\right) \quad \mathrm{mod}\ \mathfrak{M}\end{aligned}$$

which provides

$$\begin{aligned} 1+\mu(z) &= \left\{1+\frac{z_1z_2^{l-1}}{a_0}\right\}^{\frac{r}{r+s}}\left\{1+\frac{a_0^{p+r(l-1)}\left(1+\frac{z_1z_2^{l-1}}{a_0}\right)^{p+r(l-1)}z_2^{p+q+(r+s)(l-1)}}{a_0}\right\}^{\frac{r}{(r+s)^2}}\cdots\\ &= 1+\frac{r}{r+s}\frac{z_1z_2^{l-1}}{a_0}+\frac{ra_0^{p+r(l-1)-1}}{(r+s)^2}z_2^{p+q+(r+s)(l-1)}\\ &\quad +\frac{r[p+r(l-1)]a_0^{p+r(l-1)-2}}{(r+s)^2}z_1z_2^{p+q+(r+s+1)(l-1)}+\cdots \quad \mathrm{mod}\ \mathfrak{M}\end{aligned}$$

in particular, in the development of $1+\mu(z)$ the least degree in z_2 is $p+q+(r+s)(l-1)$.

By construction, the diagonal of the matrix is

$$C^{p+q+1},\dots,C^{p+q+l-1},\lambda C^{p+q+l-1},\dots,\lambda C^{p+q+l-1}$$

and the square submatrix, of order $l-1$ corresponding to the unknowns

$$b_{p+q+i},\quad i=1,\dots,l-1,$$

is diagonal because $p+q+(r+s)(l-1)>p+q+l-1$ and no term comes from $(1+\mu(z))$.
We shall show that after some linear combinations of the lines, we obtain an upper triangular matrix, which yieds $\Delta\neq 0$.
Let $(0,j)\in E_m\setminus E_{m-1}$ ($E_{-1}:=\emptyset$). Since $A_{0,j}\neq 0$, the homogeneous part of bidegree $(0,j+p+q+l-1)$ is by lemma 37, 5)

$$A_{\alpha\beta}a_0^{p\alpha+r\beta}z_2^{\alpha(p+q)+\beta(r+s)}=\lambda A_{0j}z_2^j(Cz_2)^{p+q+l-1},\quad \mathrm{mod}\ \mathfrak{M}$$

with $(\alpha,\beta)\in E_{m+1}\setminus E_m$, if such (α,β) exists, or

$$0=\lambda A_{0j}z_2^j(Cz_2)^{p+q+l-1},\quad \mathrm{mod}\ \mathfrak{M}$$

otherwise. A term $b_iz_2^i(1+\mu(z))^i$ has no part of homogeneous bidegree $(0,m)$ because $j+p+q+l-1<2(p+q)+(r+s)(l-1)$. Therefore, with the chosen order on the unknowns, all coefficients

of the linear equation are <u>over</u> the diagonal of the matrix.
Remain homogeneous parts of bidegree $(1, j+p+q+l-1)$ involving $A_{1,j}$ for $j \geq 1$. We have

$$
\begin{aligned}
(1+\mu(z))^i &= 1 + \frac{ir}{r+s}\frac{z_1 z_2^{l-1}}{a_0} + \frac{ira_0^{p+r(l-1)-1}}{(r+s)^2} z_2^{p+q+(r+s)(l-1)} \\
&\quad + \frac{ir[p+r(l-1)]a_0^{p+r(l-1)-2}}{(r+s)^2} z_1 z_2^{p+q+(r+s+1)(l-1)} + \cdots \mod \mathfrak{M}
\end{aligned}
$$

It is easy to check that for $i \geq p+q$,

$$i+p+q+(r+s+1)(l-1) > j+(p+q+l-1),$$

therefore the only terms which may be involved in homogeneous part of bidegree $(1, j+p+q+l-1)$ are

$$b_i C^i z_2^i \frac{ir}{r+s}\frac{z_1 z_2^{l-1}}{a_0}, \quad \text{where} \quad p+q \leq i \leq p+q+l-1$$

therefore

$$i = j+p+q.$$

We have still to check that $j \leq l-1$. Since $j \geq 1$, $l \geq 2$. If $(1,j) \in E_\infty$, then

$$p+q+(r+s)j \leq (p+q+l-1)\frac{r+s}{r+s-1}$$

This equivalent to

$$j \leq \frac{p+q}{(r+s-1)(r+s)} + \frac{l-1}{r+s-1}$$

As

$$\frac{p+q}{(r+s-1)(r+s)} + \frac{l-1}{r+s-1} \leq \frac{1}{r+s} + \frac{l-1}{r+s-1}$$

taking if necessary the integral part of the last member, the inequality $j \leq l-1$ is still fulfilled.

Now, there are two possibilities

1. There is no (α, β) such that $\alpha(p+q)+\beta(r+s) = j+p+q$. Therefore

   $$0 = \lambda A_{1j} z_1 z_2^j (Cz_2)^{p+q+l-1} + b_{p+q+j} C^{p+q+j} z_2^{p+q+j} \frac{(p+q+j)r}{r+s}\frac{z_1 z_2^{l-1}}{a_0} \mod \mathfrak{M}$$

   The j-th equation (which gives the j-th line L_j of the matrix) is

   $$0 = b_{p+q+j} C^{p+q+j} \mod \mathfrak{M}$$

   therefore substracting $\frac{(p+q+j)r}{a_0(r+s)} L_j$ we remove the coefficient $b_{p+q+j} C^{p+q+j} \frac{(p+q+j)r}{a_0(r+s)}$ which was under the diagonal.
2. There exists (α, β) such that $\alpha(p+q)+\beta(r+s) = p+q+j$. By lemma 37, 4), there is at most two such coefficients $(0, \beta)$ and $(1, \beta')$. By the choice of the ordering, and lemma 37, 2), $A_{1,\beta'} > A_{1j}$ and the coefficient of $A_{1,\beta'}$

   $$-a_0^{p+r\beta'-1}(p+r\beta')$$

   is over the diagonal.
   Then mod $\mathfrak{M}$, the homogeneous part of bidegree $(1, j+p+q+l-1)$ is

   $$
   \begin{aligned}
   &A_{0\beta}\, a_0^{r\beta}\, r\beta\, \frac{z_1 z_2^{l-1}}{a_0} z_2^{\beta(r+s)} + A_{1\beta'}\, a_0^{p+r\beta'}\, (p+r\beta')\, \frac{z_1 z_2^{l-1}}{a_0} z_2^{(p+q)+\beta'(r+s)} \\
   &\qquad = \lambda A_{1j} z_1 z_2^j (Cz_2)^{p+q+l-1} + b_{p+q+j} C^{p+q+j} z_2^{p+q+j} \frac{(p+q+j)r}{r+s}\frac{z_1 z_2^{l-1}}{a_0}
   \end{aligned}
   $$

hence

$$A_{0\beta}\, a_0^{r\beta-1}\, r\beta + A_{1\beta'}\, a_0^{p+r\beta'-1}\, (p+r\beta') = \lambda A_{1j} C^{p+q+l-1} + b_{p+q+j} C^{p+q+j} \frac{(p+q+j)r}{r+s} \frac{1}{a_0}$$

where perhaps one of the coefficients $A_{0\beta} = 0$ or $A_{1\beta'} = 0$. The j-th equation derived from the homogeneous part of bidegree $(0, p+q+j)$ is

$$A_{0\beta} a_0^{r\beta} + A_{1\beta'} a_0^{p+r\beta'} = b_{p+q+j} C^{p+q+j} \quad \bmod \mathfrak{M}$$

and if $A_{0\beta} = 0$, it remains to substract $\frac{(p+q+j)r}{a_0(r+s)} L_j$ to obtain a triangular matrix. If $A_{0\beta} \neq 0$, we have two coefficients under the diagonal: $A_{0\beta}$ and b_{p+q+j}. However (miracle !)

$$r\beta = \frac{(p+q+j)r}{r+s}$$

therefore substrating $\frac{(p+q+j)r}{a_0(r+s)} L_j = \frac{r\beta}{a_0} L_j$ we remove both coefficients, obtaining the desired upper triangular matrix.

We conclude that $\Delta = C^{p+q+1} \cdots C^{p+q+l-1} (\lambda C^{p+q+l-1})^{\mathrm{Card}\, E_\infty} \neq 0$. The second member of the Cramer system is nonzero and involves A_{10} and $b_{p+q} = 1$, therefore solutions of the system are rational fractions in variables $a_1, \ldots, a_{l-1}$. □

Consider the restriction of the equivalence relation defined by L. Since a_0 is fixed, lemma 4.28 shows that we have the extra condition $A = B$ and

$$a = (a_1, \ldots, a_{l-1}) \sim a' = (a'_1, \ldots, a'_{l-1}) \iff a'_i = B^i a_i, \quad \text{for} \quad i = 1, \ldots, l-1, l+K,$$

where

$$B^{k-1} = B^{r+s-1} = 1$$

and

$$B^\sigma = B^{p+q+l-1} = B^{p+q+(r+s)l-1} = 1.$$

Let $\Pi_L : \mathbb{C}^{l-1} \to \mathbb{C}^{l-1}/L$ be the canonical mapping (when there are twisted vector fields, $\sigma = (K+1)(r+s-1)$ and $L = \mathbb{Z}_{k-1}$). Similarly, consider the restriction to $\mathbb{C}^{l-1}$ of the equivalence relation of Favre germs given by lemma 3.15. We have $\epsilon^{k-1} = 1$ and if we fix λ (recall that by Prop. 35, λ depends on a_0) then $\epsilon^{p+q+l-1} = \epsilon^\sigma = 1$ and

$$b = (b_{p+q+1}, \ldots, b_{p+q+l-1}) \sim b' = (b'_{p+q+1}, \ldots, b'_{p+q+l-1}) \iff b'_{p+q+i} = \epsilon^i b_{p+q+i}, 1 \leq i \leq l-1.$$

We see that the equivalence relations $a \sim a'$ and $b \sim b'$ on $\mathbb{C}^{l-1}$ are equal.

4.1.5 Explicit construction of the isomorphic polynomial mapping (no global twisted vector fields)

In this section we show that $f_{a_0,\epsilon}$ is polynomial. We still suppose that there is .

Proposition 4. 39 *We choose $a_0 \in \mathbb{C}^\star$ and ϵ such that $\epsilon^{r+s-1} = 1$. Let $\sigma = p+q+l-1$ and suppose that $r+s-1$ does not divide $l-d$. Then there is a bijective triangular polynomial mapping*

$$\begin{array}{cccc} f_{a_0,\epsilon} : & \mathbb{C}^{l-1} & \longrightarrow & \mathbb{C}^{l-1} \\ & a = (a_1, \ldots, a_{l-1}) & \longmapsto & \Big(b_{p+q+1}(a), \ldots, b_{p+q+l-1}(a)\Big) \end{array}$$

such that for $j = 1, \ldots, l-1$,

$$b_{p+q+j}(a) = \frac{\delta\, a_j}{C^j (r+s) a_0} + R_j(a_1, \ldots, a_{j-1}),$$

$$G(z_1,z_2)=\left(\left(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}\right)^p z_2^q,\ \left(z_1z_2^l+\sum_{i=0}^{l-1}a_iz_2^{i+1}\right)^r z_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1,z_2)=\left(\lambda z_1z_2^\sigma+\sum_{i=p+q}^{\sigma}b_iz_2^i, z_2^{r+s}\right),$$

where λ depends only on a_0 by lemma 4.35.

Proof: Denote by $f_{a_0,\epsilon}$ be the rational mapping of lemma 4.38.
For $1\le j\le l-1$, the homogeneous part of bidegree $(0,p+q+j)$ is:

$$b_{p+q+j}C^{p+q+j}z_2^{p+q+j}+\sum_{j'=1}^{j-1}b_{p+q+j'}C^{p+q+j'}z_2^{p+q+j'}P_{jj'}(a_1,\ldots,a_{j-j'})z_2^{j-j'}$$

$$-\sum_{\substack{(\alpha,\beta)\neq(1,0)\\ \alpha(p+q)\\ +\beta(r+s)\\ \le p+q+j}}A_{\alpha\beta}a_0^{\alpha p+\beta r}z_2^{\alpha(p+q)+\beta(r+s)}\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1}a_iz_2^i\right\}^{\alpha p+\beta r}\right\rangle_{\left(0,p+q+j-\alpha(p+q)-\beta(r+s)\right)}$$

$$=A_{10}z_2^{p+q}\left\langle a_0^p\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1}a_iz_2^i\right\}^p\right\rangle_{(0,j)}-C^{p+q}z_2^{p+q}\Big\langle(1+\mu(z))^{p+q}\Big\rangle_{(0,j)}$$

After cancellation of z_2^{p+q+j} and recalling that $A_{10}a_0^p=C^{p+q}$, we obtain the j-th equation

$$b_{p+q+j}C^{p+q+j}+\sum_{j'=1}^{j-1}b_{p+q+j'}C^{p+q+j'}P_{jj'}(a_1,\ldots,a_{j-j'})$$

$$-\sum_{\substack{(\alpha,\beta)\neq(1,0)\\ \alpha(p+q)\\ +\beta(r+s)\\ \le p+q+j}}A_{\alpha\beta}a_0^{\alpha p+\beta r}\frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1}a_iz_2^i\right\}^{\alpha p+\beta r}\right\rangle_{\left(0,p+q+j-\alpha(p+q)-\beta(r+s)\right)}}{z_2^{p+q+j-\alpha(p+q)-\beta(r+s)}}$$

$$=C^{p+q}\left(\frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1}a_iz_2^i\right\}^p\right\rangle_{(0,j)}-\Big\langle(1+\mu(z))^{p+q}\Big\rangle_{(0,j)}}{z_2^j}\right)$$

We show that for $j=1,\ldots,l-1$,

$$b_{p+q+j}=b_{p+q+j}(a_1,\ldots,a_j)=\frac{\delta a_j}{Ca_0(r+s)}+R_j(a_1,\ldots,a_{j-1})\quad\text{with } R_j\in\mathbb{C}[a_1,\ldots,a_{j-1}].$$

In fact

- For $j'=1,\ldots,j-1$, $P_{jj'}\in\mathbb{C}[a_1,\ldots,a_{j-1}]$,
- Since $p+q+j-\alpha(p+q)-\beta(r+s)<j$,

$$\frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1}a_iz_2^i\right\}^{\alpha p+\beta r}\right\rangle_{\left(0,p+q+j-\alpha(p+q)-\beta(r+s)\right)}}{z_2^{p+q+j-\alpha(p+q)-\beta(r+s)}}\in\mathbb{C}[a_1,\ldots,a_{j-1}],$$

- Clearly *mod* $\mathfrak{M}_{j-1} := \big(a_1, \ldots, a_{j-1}\big)$,

$$\frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{l-1} a_i z_2^i\right\}^p\right\rangle_{(0,j)}}{z_2^j} = \frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{j} a_i z_2^i\right\}^p\right\rangle_{(0,j)}}{z_2^j} = \frac{pa_j}{a_0}$$

- From the definition of μ, *mod* $\mathfrak{M}_{j-1}$,

$$\frac{\left\langle(1+\mu(z))^{p+q}\right\rangle_{(0,j)}}{z_2^j} = \frac{\left\langle\left\{1+\frac{1}{a_0}\sum_{i=1}^{j} a_i z_2^i\right\}^{\frac{r(p+q)}{r+s}}\cdots\right\rangle}{z_2^j} = \frac{r(p+q)a_j}{(r+s)a_0}$$

Therefore modulo $\mathfrak{M}_{j-1}$,

$$b_{p+q+j}C^j = \frac{\delta\, a_j}{(r+s)a_0}$$

and there exists a polynomial $R_j(a_1, \ldots, a_{j-1}) \in \mathbb{C}[a_1, \ldots, a_{j-1}]$ without constant term such that

$$(T) \qquad b_{p+q+j} = b_{p+q+j}(a_1, \ldots, a_j) = \frac{\delta\, a_j}{C^j(r+s)a_0} + R_j(a_1, \ldots, a_{j-1}).$$

□

Corollary 4. 40 *We choose $a_0 \in \mathbb{C}^\star$, ϵ such that $\epsilon^{r+s-1} = 1$. Let $\sigma = p+q+l-1$ and suppose that $r+s-1$ does not divide $l-d$. Then*

$$\begin{array}{ccc} \mathbb{C}^{l-1} & \xrightarrow{f_{a_0,\epsilon}} & \mathbb{C}^{l-1} \\ & \searrow^{\Pi_L} \quad {}^{\Pi_L}\swarrow & \\ & \mathbb{C}^{l-1}/\mathbb{Z}_{k-1} & \end{array}$$

where

$$f_{a_0,\epsilon} : \mathbb{C}^{l-1} \to \mathbb{C}^{l-1}, \quad (a_1, \ldots, a_{l-1}) \mapsto (b_{p+q+1}, \ldots, b_{p+q+l-1})$$

is a commutative diagram and $f_{a_0,\epsilon}$ is an isomorphic polynomial mapping.

4.1.6 Explicit construction of the isomorphic polynomial mapping (there exists global twisted vector fields)

We suppose that $l - d = K(r+s-1)$ i.e. there are non trivial global twisted vector fields. We have

$$l + K = d + Kk(S), \quad \sigma = p+q+l-1 = (k(S)-1)(K+1), \quad \frac{\sigma k(S)}{k(S)-1} = k(S)(K+1).$$

We denote by

$$\left(\sum\right) := \left(\sum_{i=1}^{l-1} a_i z_2^i + a_{l+K}\, z_2^{l+K} + z_1 z_2^{l-1}\right)$$

the equation (I) is now

$$(I) \quad \begin{cases} \varphi_1\left(a_0^p\left\{1+\frac{1}{a_0}\left(\sum\right)\right\}^p z_2^{p+q}, a_0^r\left\{1+\frac{1}{a_0}\left(\sum\right)\right\}^r z_2^{r+s}\right) \\ \\ \quad = \lambda\varphi_1(z)\Big(Cz_2(1+\mu(z)\Big)^\sigma + \sum_{i=p+q}^{\sigma} b_i\Big(Cz_2(1+\mu(z))\Big)^i + c\Big(Cz_2(1+\mu(z))\Big)^{\frac{\sigma k}{k-1}} \end{cases}$$

If $m \geq \frac{\sigma}{k-1}$, then $(0,m) \notin E_\infty$, in fact

$$m(r+s) \geq \frac{\sigma k}{k-1} = (p+q+l-1)\frac{r+s}{r+s-1},$$

therefore the coefficients a_{l+K} and c doesn't occur in the previous calculations and we obtain by similar arguments a polynomial mapping $f_{a_0,\epsilon}$.

Lemma 4. 41 *Suppose $l = d + K(r+s-1)$. Let $\mathfrak{M} = \mathfrak{M}_{l-1} = (a_1, \ldots, a_{l-1})$ and (i,j) such that*

$$\left\langle A_{ij} a_0^{pi+rj} \left\{1 + \tfrac{1}{a_0}\left(\textstyle\sum_{m=1}^{l-1} a_m z_2^m + a_{l+K} z_2^{l+K} + z_1 z_2^{l-1}\right)\right\}^{pi+rj} z_2^{i(p+q)+j(r+s)} \right\rangle_{(0,\frac{\sigma k}{k-1})} \neq 0, \mod \mathfrak{M}$$

then, $(i,j) = (1,0)$ or $(i,j) = (0, \frac{\sigma}{k-1})$. More precisely homogeneous part of bidegree $(0, \frac{\sigma k}{k-1})$ is

$$c\, C^{\frac{\sigma k}{k-1}} = A_{10}\, p a_0^{p-1} a_{l+K} + A_{0,\frac{\sigma}{k-1}} C^\sigma (1-\lambda) \mod \mathfrak{M}.$$

In particular if there are global vector fields, i.e. $\lambda = 1$,

$$c\, C^{\frac{\sigma k}{k-1}} = A_{10}\, p a_0^{p-1} a_{l+K} \mod \mathfrak{M}.$$

Proof: 1) If $(i,j) \in E_\infty$, then $i(p+q) + j(r+s) < \frac{\sigma k}{k-1}$ and $i \leq 1$ by lemma 4.37 4).

- Case $i = 1$: Since
$$l + K + (p+q) + j(r+s) = \frac{\sigma k}{k-1} + jk \geq \frac{\sigma k}{k-1}$$
we have equality if $j = 0$ hence $(i,j) = (1,0)$.
- Case $i = 0$: then $1 \leq j < \frac{\sigma}{k-1}$ and mod $\mathfrak{M}$,
$$\left\langle A_{0j} a_0^{rj} \left\{1 + \frac{a_{l+K} z_2^{l+K}}{a_0}\right\}^{rj} z_2^{j(r+s)} \right\rangle_{(0,\frac{\sigma k}{k-1})} \neq 0, \mod \mathfrak{M}$$
In the left member the possible powers of z_2 are of the form $\alpha(l+K) + jk$ with $\alpha \geq 0$ and $j \geq 1$ such that
$$\alpha(l+K) + jk = \frac{\sigma k}{k-1}.$$
Since $\alpha(l+K) + jk = (d+Kk)\alpha + jk$, we derive that $\alpha \geq 1$ is impossible, therefore $\alpha = 0$ and $j = \frac{\sigma}{k-1}$.

2) From 1) we deduce that there exists a polynomial P in variables $a_1, \ldots, a_{l-1}$ such that the coefficients of $z_2^{\frac{\sigma k}{k-1}}$ in (I) give the equality

$$A_{10}\, p a_0^{p-1} a_{l+K} + A_{0,\frac{\sigma}{k-1}} a_0^{\frac{\sigma r}{k-1}} = \lambda A_{0\frac{\sigma}{k-1}} C^\sigma + c\, C^{\frac{\sigma k}{k-1}} + P(a_1, \ldots, a_{l-1}).$$

By equation (5),

$$a_0^{\frac{\sigma r}{k-1}} - \lambda C^\sigma = C^\sigma(1-\lambda)$$

which gives the result. □

Proposition 4. 42 *If $l - d = K(r+s-1)$ and $\lambda = 1$, there is a bijective triangular polynomial mapping*

$$\begin{array}{rccc} g_{a_0,\epsilon}: & \mathbb{C}^{l-1} \times \mathbb{C} & \longrightarrow & \mathbb{C}^{l-1} \times \mathbb{C} \\ & a = (a_1, \ldots, a_{l-1}, a_{l+K}) & \longmapsto & \Big(b_{p+q+1}(a), \ldots, b_{p+q+l-1}(a), c(a)\Big) \end{array}$$

such that,

$$b_{p+q+j}(a) = \frac{\delta\, a_j}{C^j(r+s)a_0} + R_j(a_1,\dots,a_{j-1}), \quad j=1,\dots,l-1,$$

$$c(a) = C^{-\frac{\sigma k}{k-1}} A_{10}\, p a_0^{p-1} a_{l+K} + R(a_1,\dots,a_{l-1}),$$

$$G(z_1,z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^p z_2^q,\ \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{l+K+1}\right)^r z_2^s\right)$$

is conjugated to the polynomial germ

$$F(z_1,z_2) = \left(\lambda z_1 z_2^\sigma + \sum_{k=p+q}^{\sigma} b_k z_2^k + c z_2^{\frac{\sigma k(S)}{k(S)-1}}, z_2^{r+s}\right).$$

Proof: We have a bijective polynomial map

$$f_{a_0,\epsilon} : \mathbb{C}^{l-1} \to \mathbb{C}^{l-1}, \quad a \mapsto b = f_{a_0,\epsilon}(a).$$

From lemma 5.41, when $a = (a_1,\dots,a_{l-1})$ is fixed and $a_{l+K} \in \mathbb{C}$, the mapping $c : \mathbb{C} \to \mathbb{C}$, $a_{l+K} \mapsto c = c(a_{l+K}) = C^{-\frac{\sigma k}{k-1}} A_{10}\, p a_0^{p-1} a_{l+K}$ is linear hence bijective.
$\square$

Corollary 4. 43 *Any surface with GSS with one branch admits a special birational structure.*

Corollary 4. 44 *The intersection $A := Aut(\mathbb{C}^2, H, 0) \cap \Phi$ is the trivial group or a group isomorphic to $(\mathbb{C},+)$. Moreover*

- *if $k-1$ does not divide $\mathfrak{s} = p+q+l-1$, the canonical mapping*

  $$g : \mathcal{G}/A = \mathcal{G}(p,q,r,s,l)/A \to U_{k,\mathfrak{s},m_1}/\mathbb{Z}_{k-1}$$

  to the Oeljeklaus-Toma coarse moduli space of marked surfaces (S, C_0) with one branch

  $$U_{k,\mathfrak{s},m_1}/\mathbb{Z}_{k-1} = \mathbb{C}^\star \times \mathbb{C}^{l-1}/\mathbb{Z}_{k-1}$$

  is isomorphic and there is a polynomial lifting

  $$(\lambda, b) : \mathbb{C}^\star \times \mathbb{C}^{l-1} \to \mathbb{C}^\star \times \mathbb{C}^{l-1}$$

  which is a covering such that

  $$\begin{tikzcd} \mathbb{C}^\star \times \mathbb{C}^{l-1} \arrow[r, "(\lambda,b)"] \arrow[d] & \mathbb{C}^\star \times \mathbb{C}^{l-1} \arrow[d] \\ \mathcal{G}/A \arrow[r, "g"] & U_{k,\mathfrak{s},m_1}/\mathbb{Z}_{k-1} \end{tikzcd}$$

  is commutative,

- *if $k-1$ divides $\mathfrak{s} = p+q+l-1$, we have similar results for*

  $$U^{\lambda\neq 0, c=0}_{k,\mathfrak{s},m_1}/\mathbb{Z}_{k-1} \quad \text{and} \quad U^{\lambda=1}_{k,\mathfrak{s},m_1}/\mathbb{Z}_{k-1}.$$

Corollary 4. 45 *Let $\mathcal{S}_{J,\sigma} \to B_J$ be a large family with $\sigma = Id$. Let $T_{J,\sigma}$ the hypersurface where cocycles $[\theta^i]$ and $[\mu^i]$ are not independent. Then for each stratum $B_{J,M}$, the trace $T_{J,\sigma} \cap B_{J,M}$ on $B_{J,M}$ is equal to the inverse image of the ramification set by the lift of the canonical mapping i.e.*

- *If $k-1$ does not divide $\mathfrak{s}$,*

  $$T_{J,\sigma} \cap B_{J,M} = (\lambda, b)^{-1}(T_{k,\mathfrak{s},m_1}),$$

- *If $k-1$ divides $\mathfrak{s}$*

  $$T_{J,\sigma} \cap B_{J,M} = (\lambda, b)^{-1}(T^{\lambda\neq 1, c=0}_{k,\mathfrak{s},m_1}).$$

In particular in B_J there is no curve over which the surfaces are isomorphic.

4.2 Special birational structures on Kato surfaces

Let S be a Kato surface with $b_2(S) = n$, let $D_0, \dots, D_{n-1}$ be its rational curves, $p : \tilde S \to S$ its universal cover, C_0 a lift of D_0, C_i, $i \in \mathbb{Z}$, the rational curves in $\tilde S$ in the canonical order and $\mathfrak{U} = (U_i)_{0\le i\le n-1}$ an Enoki covering of S such that U_0 contains C_0 with a deleted disc (see the construction of these surfaces). If S is associated to a germ $F = \Pi\sigma$ where σ is birational, then S is endowed with a birational structures as well as $\tilde S$ and p is a $\big(Bir(\mathbb{P}^2(\mathbb{C})), \mathbb{P}^2(\mathbb{C})\big)$-morphism.

Definition 4. 46 *A birational structure on a surface with GSS S will be called* **special** *if there is a contracting germ $F = \Pi\sigma$ with F or equivalently σ birational and $S = S(F)$.*

This definition is independent of the numbering. Each open set U_i is covered by two charts U_i' and U_i'' with local coordinates $\varphi_i' = (u_i', v_i') : U_i' \to \mathbb{C}^2$ and $\varphi_i'' = (u_i'', v_i'') : U_i'' \to \mathbb{C}^2$ respectively. We denote by $\varphi_i = (u_i, v_i)$ the local coordinates whose domain contains the blown-up point $O_i \in C_i$. Here $\varphi_i = Id$. Then, with the identification

$$i : \mathbb{C}^2 \simeq \{[z_0 : z_1 : z_2] \in \mathbb{P}^2(\mathbb{C}) \mid z_2 = 1\} \subset \mathbb{P}^2(\mathbb{C}),$$

$$b_{i,i+1}(u_{i+1}, v_{i+1}) = \varphi_i \circ \varphi_{i+1}^{-1}(u_{i+1}, v_{i+1}) = \Pi_{i+1}(u_{i+1}, v_{i+1}),\ i = 0, \dots, n-2, \quad \beta_{n-1,0}(u_0, v_0) = \sigma \circ \Pi_0(u_0, v_0).$$

If S contains a GSS but is not minimal the order on the curves is no more total.

Lemma 4. 47 *Let $S = S(\Pi, \sigma)$ be a surface containing a GSS (not necessarily minimal) such that $n = b_2(S)$. If σ is birational, there exist for any $j \in \mathbb{Z}$*

- *a meromorphic developing map $\widehat{Dev}_j : \hat S_{C_j} \to \mathbb{P}^2$, locally biholomorphic outside the rational curves, such that $\widehat{Dev}_j(C_j)$ is the rational curve $\{z_1 = 0\} \subset \mathbb{P}^2(\mathbb{C})$ and $O_j := \widehat{Dev}_j(\hat O_{C_j}) = [a_j : b_j : 1]$,*
- *a birational mappings $G_j : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$, holomorphic in a neighbourhood of $\hat O_{C_j}$.*

such that the following diagrams are commutative:

$$\begin{tikzcd}
\tilde S \arrow[rr, "\tilde g"] \arrow[d, "p_{C_j}"'] & & \tilde S \arrow[d, "p_{C_{j+n}}"] \\
\hat S_{C_j} \arrow[rr, "\sigma_j^{j+n}"] \arrow[dr, "\widehat{Dev}_j"'] & & \hat S_{C_{j+n}} \arrow[dl, "\widehat{Dev}_{j+n}"] \\
& \mathbb{P}^2(\mathbb{C}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde S \arrow[r, "\tilde g"] \arrow[d, "p_{C_j}"'] & \tilde S \arrow[d, "p_{C_j}"] \\
\hat S_{C_j} \arrow[r, "F_{C_j}"] \arrow[d, "\widehat{Dev}_j"'] & \hat S_{C_j} \arrow[d, "\widehat{Dev}_j"] \\
\mathbb{P}^2(\mathbb{C}) \arrow[r, "G_j"] & \mathbb{P}^2(\mathbb{C})
\end{tikzcd}$$

Moreover if S is minimal (i.e. if S is a Kato surface) then O_j is a fixed point of G_j.

Proof: To simplify the notations we may suppose that $j = 0$. Recall that by construction $W_0 \subset \hat S_{C_0}$ and then we apply lemma 2.7. Since all birational transition functions are isomorphic outside the curves, the developing map $\widehat{Dev}_0$ is a local biholomorphism outside the rational curves. We denote a blowup $\Pi_{i+1} : U_{i+1} \to B_i \subset W_i$ by

$$\Pi_{i+1} : (u_{i+1}', v_{i+1}') \mapsto (u_{i+1}' v_{i+1}' + a_i, v_{i+1}') = (u_i, v_i),$$

$$\Pi_{i+1} : (u_{i+1}'', v_{i+1}'') \mapsto (v_{i+1}'' + a_i, u_{i+1}'' v_{i+1}'') = (u_i, v_i)$$

with inverse

$$\Pi_{i+1}^{-1} : (u_i, v_i,) \mapsto (u_{i+1}', v_{i+1}') = \left(\frac{u_i - a_i}{v_i}, v_i\right),$$

$$\Pi_{i+1}^{-1} : (u_i, v_i,) \mapsto (u''_{i+1}, v''_{i+1}) = \left(\frac{v_i}{u_i - a_i}, u_i - a_i\right)$$

On U_0, $\widehat{Dev}_0$ is defined in the following way: If $\hat{O}_{C_0}$ is in the chart (u'_0, v'_0),

$$\widehat{Dev}_0(u'_0, v'_0) = [u'_0 : v'_0 : 1], \quad \widehat{Dev}_0(u''_0, v''_0) = [\frac{1}{u''_0} : u''_0 v''_0 : 1] = [1 : u''^2_0 v''_0 : u''_0].$$

The construction is similar if $\hat{O}_{C_0}$ is in the chart (u''_0, v''_0).
On $\bigcup_{k<0} U_k$, we have

$$\begin{array}{rccl} \widehat{Dev}_0 : & U_k & \to & \mathbb{P}^2(\mathbb{C}) \\ & (u_k, v_k) & \mapsto & \widehat{Dev}_0(u_k, v_k) = i \circ b_{0,-1} \circ \cdots \circ b_{k+1,k}(u_k, v_k) \end{array}$$

i.e. for $k > -n - 1$,

$$\widehat{Dev}_0(u_k, v_k) = i \circ \Pi_0^{-1} \circ \sigma_0^{-1} \circ \Pi_{-1}^{-1} \circ \cdots \circ \Pi_{k+1}^{-1}(u_k, v_k)$$

and $\sigma_0\Pi_0 : U_0 \to U_{-1}$ is the composition of $\Pi_0 : U_0 \to B$ and of $\sigma_0 : B \to U_{-1}$ which is birational induced by σ. The image $\sigma_0\Pi_0(U_0)$ is a ball in W_{-1}; $\sigma_0\Pi_0$ being birational $\Pi_0^{-1}\sigma_0^{-1}$ extends to U_{-1}.
The points $(a_k, 0) \in W_k$ are indeterminacy points of Π_{k+1}^{-1} however do not belong to U_k. Therefore $\Pi_{k+1}^{-1}(U_k)$ has an empty intersection with C_{k+1} and $\widehat{Dev}_0$ is holomorphic.
The upper parts of the diagrams are commutative by [3], p30; to see the commutativity of the lower parts it is sufficient to check it on the chart (u_0, v_0). $\square$ The following theorem shows that we recover a GSS in S thanks to a small sphere centered at $\widehat{Dev}_j(\hat{O}_j)$.

Theorem 4. 48 *Let $S = S(\Pi, \sigma)$ be a Kato surface such that $n = b_2(S)$. If σ is birational, there exist for any $j \in \mathbb{Z}$ a meromorphic developing map $\widetilde{Dev}_j : \tilde{S} \to \mathbb{P}^2$, locally biholomorphic outside the rational curves, such that $\widetilde{Dev}_j(C_j)$ is the rational curve $\{z_1 = 0\} \subset \mathbb{P}^2(\mathbb{C})$ and $O_j := \widetilde{Dev}_j(p_{C_j}^{-1}(\hat{O}_{C_j})) = [a_j : b_j : 1]$, i.e. $\widetilde{Dev}_j$ blows down an infinite number of curves. Moreover for any small ball B_j centered at O_j, $p(\widetilde{Dev}_j^{-1}(\partial B_j))$ is a GSS in S.*

Proof: $\widetilde{Dev}_j = \widehat{Dev}_j \circ p_{C_j}$ has the expected properties. $\square$

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